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THEORETICAL STUDY OF SPACE PLASMAS

Final Report

February 16, 1964 to March 15, 1965

by

D. B. Chang, L. D. Pearlstein, and M. N. Rosenbluth

Prepared for
National Aeronautics and Space Administration
Office of Space Sciences, Geophysics and Astronomy Programs
under Contract NASw-907

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PART 1

The final report under Contract NASw-907 consists of three papers (GA-6007, GA-5891, GA-6124) which have been submitted for publication. These papers comprise Parts 2-4, respectively. In this part we summarize the conclusions arrived at in these reports. Finally, Part 5 lists the activities pursued by the program personnel during the last four months of the contract period.

In Part 2 we examine the stability of the Van Allen belt to spontaneous interchange instabilities and conclude that with a reasonable choice of parameters the belt is conclusively stable during the day; however, with nighttime ionospheric parameters the radiation belt is on the border of instability and we conjecture that the distribution of particles in the belt may be limited by the nighttime ionospheric parameters.

In Part 3 we examine the effect of resonant particle — Whistler and Alfvén wave interaction, a magnetic moment violating process. We show that this process is an effective dumping mechanism; in addition, those particles which survive are left in a flat helix distribution.

Finally, in Part 4 we look at an exact solution of the "Universal" instability in the drift approximation and demonstrate that the usual results obtained from a local treatment of the differential equation is consistent with the exact dispersion relation provided the standard conditions are satisfied, viz., the density varies slowly enough so that a WKB treatment is valid.

GA-6007

N66 11140

PART 2

ON THE INTERCHANGE STABILITY OF THE VAN ALLEN BELT

BY

D. B. Chang, L. D. Pearlstein, and M. N. Rosenbluth*

This is a preprint of a paper submitted by the authors for publication in the Journal of Geophysical Research.

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ABSTRACT 11140

The interchange stability of the Van Allen belt is studied in the electrostatic approximation. The destabilizing currents are obtained from the description of the particle motion developed by Northrup and Teller (1960) when the magnetic moment and bounce actions are invariant. The stabilizing currents are described by the three fluid (electrons, ions and neutrals) zero temperature dielectric tensor discussed by Fejer (1960). For frequencies of the order of the azimuthal drift periods of the energetic particles in the belt, the stabilizing currents arise primarily from the ionosphere. With a reasonable choice of belt parameters, the energetic particles appear to be quite stable against interchanges for the electron densities which obtain in the ionosphere during the daytime, but are near marginal stability for nighttime ionospheric densities. Author

I. INTRODUCTION

The possible importance of large-scale motions in the magnetosphere of the earth in which the plasmas in tubes of equal flux are interchanged, has been discussed by Gold (1959, 1962), Axford and Hines (1961) and Sonnerup and Laird (1963). These authors have investigated the change of the potential energy of the system due to an interchange of two neighboring tubes of force, and have found that under some circumstances the energy of the plasma treated as a simple fluid decreases. They point out that if these energetically allowable interchanges occur, a major mechanism will result for the redistribution of energetic charged particles in the magnetosphere.

In the following we study the dynamics of the interchanges further to see if energetically allowable ones can actually occur spontaneously. In Section II, the normal mode equation for electrostatic interchanges is obtained. There the destabilizing currents are obtained from the description of the particle motion developed by Northrup and Teller (1960) when the magnetic moment and bounce actions are invariant, while the stabilizing currents are described by a zero temperature three fluid (electrons, ions and neutrals) dielectric tensor which has been discussed in detail by Fejer (1960). In Section III the significance of the results for the radiation belt is discussed.

II. DISPERSION FORMULA FOR INTERCHANGES

A dispersion formula for permissible motions is obtained on combining Maxwell's equations with the equations governing the particle motions, requiring that the electromagnetic fields are such that in the particle equations they induce currents and charge densities which in Maxwell's equations are consistent with the fields. In order for an interchange motion to result in a net change in the distribution of the energetic radiation belt particles, the motion must violate one or more of the three adiabatic invariants of these particles. In the following we shall consider motions which have time scales longer than the longitudinal mirroring times but possibly less than the azimuthal drift times of the energetic particles. The third (or azimuthal) adiabatic invariant is therefore possibly violated while the first Larmor and second (mirror) adiabatic invariant are not violated. The particle drift motions in the earth's field under these circumstances have been discussed in detail by Northrup and Teller (1960), and we shall use their equations to describe the motion of the high-energy particles. Similar equations have been used in discussing the energetics of "minimum B" confinement schemes in the work on controlled fusion (J. B. Taylor - 1963).

i. Particle Motion

Northrup and Teller start with the invariance of the quantity

$$\mu = p_{\perp}^2/2m_o B \tag{1}$$

and the guiding center equations of motion

$$\frac{dp_{\parallel}}{dt} = -\frac{\mu}{\gamma}\frac{dB}{\partial s} + e\vec{E}\cdot\vec{b} \tag{2}$$

$$\vec{u}_d = \frac{\vec{b}}{B} \times \left(-c\vec{E} + \frac{\mu c}{\gamma e}\nabla B + \frac{c}{\gamma e}\frac{p_{\parallel}^2}{m_o}\frac{d\vec{b}}{\partial s} \right) \tag{3}$$

which result after an average of the equation of motion of a particle of change e has been taken over the rapid gyration around a field line. Here $\vec{B}$ and $\vec{E}$ are the magnetic and electric fields, $p_{\perp}$ and $p_{\parallel}$ are the components of the particles' relativistic momentum $\vec{p}$ perpendicular and parallel to $\vec{B}$, m_o is the rest mass; γ equals the total mass divided by m_o; $\vec{b}$ is the unit vector $\vec{B}/B$ along the line of force, s is the distance along the line of force, $\vec{u}_d$ is the drift velocity which moves the guiding center to a neighboring line, c is the speed of light, and t denotes the time. By averaging the guiding center equations of motion over the longitudinal oscillation between mirroring points, they then find on using the invariance of μ the second invariant

$$J = \oint p_{\parallel} ds, \tag{4}$$

and the following equations describing the average behavior in a mirror period:

$$\langle\dot{\alpha}\rangle = -\frac{c}{e}\frac{\partial K}{\partial \beta}(\alpha, \beta, J, \mu, t) \tag{5}$$

$$\langle\dot{\beta}\rangle = \frac{c}{e}\frac{\partial K}{\partial \alpha} \tag{6}$$

$$\langle\dot{K}\rangle = \frac{\partial K}{\partial t} \tag{7}$$

$$1 = T\frac{\partial K}{\partial J} . \tag{8}$$

In equation (4), the integral is taken over a complete oscillation of period T along the line. In equations (5) - (8), the variables α and β which have been introduced are such that the vector potential $\vec{A}$ is given by

$$A = \alpha\nabla\beta; \tag{9}$$

this makes

$$\vec{B} = \nabla\alpha \times \nabla\beta \tag{10}$$

and makes $d\alpha d\beta$ equal to the flux through a surface element. The quantity K is defined by

$$K = (p^2c^2 + m_o^2c^4)^{\frac{1}{2}} + e(\phi + \psi) \tag{11}$$

where ϕ is the electrostatic potential and

$$\psi = \frac{\alpha}{c}\frac{\partial\beta}{\partial t} \tag{12}$$

In equations (5) - (8), the defining relations of equations (1) and (4) must be used to eliminate $p_{\parallel}$ and $p_{\perp}$ from equation (11), so that K is expressed as a function of α, β, J, μ and t. In these equations, $\langle\dot{\alpha}\rangle$, $\langle\dot{\beta}\rangle$ and $\langle\dot{K}\rangle$ denote the average time rates of change of α, β and K experienced by a particle during a mirror oscillation.

Denoting by $F(\mu, J, \alpha, \beta, t)$ $d\mu dJ d\alpha d\beta$ the number of particles in the interval $d\mu$ at μ, dJ at J, $d\alpha$ at α, and $d\beta$ at β at time t, and forming a continuity equation for $F(\mu, J, \alpha, \beta, t)$ with μ and J invariants,

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial \alpha}\left[\langle\dot{\alpha}\rangle F\right] + \frac{\partial}{\partial \beta}\left[\langle\dot{\beta}\rangle F\right] = 0, \tag{13}$$

Northrup and Teller then point out that F satisfies a Liouville equation

$$\frac{\partial F}{\partial t} - \frac{c}{e}\left[\frac{\partial F}{\partial \alpha}\frac{\partial K}{\partial \beta} - \frac{\partial F}{\partial \beta}\frac{\partial K}{\partial \alpha}\right] = 0 \tag{14}$$

with $\frac{c}{e}K(\alpha, \beta, J, \mu, t)$ playing the role of the Hamiltonian. Finally, denoting by $f(\mu, J, \vec{x}, t)\, d\mu dJ\, d^3x$ the number of particles in the volume element d^3x at $\vec{x}$ at time t, this distribution function is related to the distribution function F for the line passing through $\vec{x}$ by

$$f = (2B/V_{\parallel}T)F, \tag{15}$$

where the factor B comes from the Jacobian relating the cross sectional area of a tube of flux at $\vec{x}$ to $d\alpha d\beta$, and $2/V_{\parallel}T$ comes from the fact that the fraction of time spent by a particle in an element ds of the line of force is simply

$$\frac{ds}{V_{\parallel}} \Big/ \oint \frac{ds}{V_{\parallel}} = \frac{2ds}{TV_{\parallel}} \quad ,$$

where $V_{\parallel} = P_{\parallel}/m_0\gamma$ is the particle's longitudinal velocity.

ii. Dispersion Relation

In a plasma where the particle motions are described by the preceding equations, we may obtain a dispersion relation for electromagnetic disturbances with time scales longer than the mirror periods of the particles by forming current and charge densities from equations (5) - (15) and inserting these in Maxwell's equations. With $\vec{B} = \nabla\alpha \times \nabla\beta$, the equation

$$\nabla \cdot \vec{B} = 0 \tag{16}$$

is automatically satisfied, and with

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial t}(\alpha\nabla\beta), \tag{17}$$

the Maxwell equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{c\partial t} \tag{18}$$

is automatically satisfied.

In terms of α, β and Φ, we then have for

$$\nabla \times \vec{B} = 4\pi\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}(\overleftrightarrow{\epsilon}\cdot\vec{E})$$

the equation

$$\nabla \times (\nabla\alpha \times \nabla\beta) + \frac{1}{c^2}\frac{\partial}{\partial t}\left[\overleftrightarrow{\epsilon}\cdot\frac{\partial}{\partial t}(\alpha\nabla\beta)\right] + \frac{1}{c}\frac{\partial}{\partial t}(\overleftrightarrow{\epsilon}\cdot\nabla\Phi) = 4\pi\vec{j}; \tag{19}$$

and for

$$\nabla \cdot (\overleftrightarrow{\epsilon}\cdot\vec{E}) = 4\pi\rho,$$

we have

$$\nabla \cdot (\overleftrightarrow{\epsilon}\cdot\nabla\Phi) + \nabla \cdot \left[\frac{1}{c}\frac{\partial}{\partial t}(\overleftrightarrow{\epsilon}\cdot\alpha\nabla\beta)\right] = -4\pi\rho. \tag{20}$$

In equations (19) and (20), ρ and $\vec{j}$ are the charge density and current density, respectively, and are given in terms of the f of equation (15) by

$$\rho(\vec{x},t) = \Sigma e \iint d\mu dJ \; f(\mu,\ J,\ \vec{x},\ t) \tag{21}$$

$$\vec{j}(\vec{x},t) = \Sigma e \iint d\mu dJ \; f(\mu,\ J,\ \vec{x},\ t)\ \vec{u}_d + \nabla\times\vec{M}\ . \tag{22}$$

The sum is over particle species, $\nabla\times\vec{M}$ is the magnetication current obtained from

$$\vec{M} = \Sigma \iint d\mu dJ \; f(\mu,J,\vec{x},t)(\frac{\mu}{\gamma}), \tag{23}$$

$\vec{u}_d$ is the drift velocity of equation (3), γ is the ratio of the mass to the rest mass, and both $\vec{u}_d$ and γ are expressed in terms of $\Phi, \alpha,\ \beta,\ \mu,\ J,\ s$ and t. In equation (23) the factor γ^{-1} is required since relativistically the magnetic moment is not invariant but is equal to the invariant μ multiplied by γ^{-1}.

The dielectric tensor $\overleftrightarrow{\epsilon}$ which occurs in the foregoing is included to describe all other relevant currents besides those resulting from the magnetic moment of equation (1) and the lowest order guiding center motions of equations (2) and (3). Since we are interested in frequencies of the order of the azimuthal drift frequencies of the energetic particles in a field configuration where the radius of curvature is of the order of the size of the system, it may be verified that the higher order finite Larmor radius corrections to the guiding center motions are not important [Rosenbluth, Krall, Rostoker (1962)]. For the present problem, then, the dielectric tensor needs only include polarization and resistive currents.

The discussion of interchange instability corresponds to an approximate solution of the foregoing equations for perturbations about equilibria where the plasma energy density is much smaller than the magnetic energy density. With only a relatively much smaller particle energy density available to create a change in the magnetic field energy, it might be expected that the fastest growing instabilities in these cases would be ones in which the magnetic field remains practically unperturbed. We are then led to consider the approximation in which the perturbed electric field is derived completely from the scalar potential Φ. In this approximation the dispersion relation is obtained from Poisson's equation,

$$\nabla\cdot(\overleftrightarrow{\epsilon}\cdot\nabla\Phi) = -4\pi\rho, \tag{24}$$

where again the charge density $\rho(\vec{x},t)$ is given by equation (21), and where here we shall regard no electric field to exist in equilibrium.

Consider now the features of the interchange instability described by equation (25) for perturbations about an azimuthally symmetric equilibrium in which the magnetic field has no azimuthal component. At any point in the field introduce a coordinate system defined by the orthogonal unit vectors $\{\vec{i}_\alpha, \vec{i}_\beta, \vec{i}_\chi\}$, where $\vec{i}_\chi$ is in the direction of the equilibrium magnetic field $\vec{B}$ at the point, $\vec{i}_\beta$ is the azimuthal direction about the axis of symmetry of the system, and $\vec{i}_\alpha = \vec{i}_\beta \times \vec{i}_\chi$ is perpendicular to the field and lies along the radius of curvature of the field line. The length of a differential line element $d\vec{\ell}$ is therefore given by

$$(d\mathscr{s})^2 = (dx_\alpha)^2 + (dx_\beta)^2 + (dx_\chi)^2 \tag{26}$$

where dx_α is the projection of the differential element along $\vec{i}_\alpha$, dx_β is the projection along $\vec{i}_\beta$, and ds is the projection along the field line. In terms of a cylindrical coordinate system (r,θ,z) chosen so that the z-axis lies along the axis of symmetry of the system, we may write

$$dx_\beta = r\,d\theta \ . \tag{27}$$

For this cylindrically symmetric system the α and β of equations (9) and (10) may be introduced by setting

$$d\alpha = Br\,dx_\alpha$$

$$d\beta = d\theta,$$

and for $\nabla\times\vec{B}\approx 0$, we may set B equal to the gradient of the magnetic scalar potential χ,

$$B = \frac{\partial\chi}{\partial s}.$$

Equation (26) may then be written

$$(d\mathscr{s})^2 = \frac{1}{(Br)^2}\,(d\alpha)^2 + r^2(d\beta)^2 + \frac{(d\chi)^2}{B^2}\ , \tag{26a}$$

and all equations involving the gradient operator ∇ may be expressed in terms of $\frac{\partial}{\partial\alpha}$, $\frac{\partial}{\partial\theta}$, and $\frac{\partial}{\partial\chi}$ by inspection.

Poisson's equation becomes

$$\nabla\cdot\left[\overleftrightarrow{\epsilon}\cdot\nabla\Phi\right] = -4\pi\rho = -4\pi\Sigma\ eB\iint d\mu dJ\ \left(\frac{2}{V_{\parallel}T}\right)F, \tag{25a}$$

where the second equality results on using equations (15) and (21). Since the particles can move very readily along the magnetic field lines to cancel out any electric fields parallel to $\vec{B}$, we may set

$$\frac{\partial \Phi}{\partial \chi} = 0 \tag{28}$$

in equation (25a). We show in the appendix that this is an equally valid assumption in the ionosphere. With Φ, α and β constant along a line, it is then reasonable to divide equation (25a) by B and integrate over s (i.e., integrate equation (25a) over χ). Since $T \equiv \oint \frac{ds}{V_\parallel}$, this operation yields

$$\oint \nabla \cdot \left[\overleftrightarrow{\epsilon} \cdot \nabla \Phi\right] \frac{ds}{B} = -8\pi \Sigma e \iint d\mu dJF. \tag{29}$$

On Fourier analyzing the perturbed quantities,

$$\Phi(\alpha,\theta) = \sum_{m=-\infty}^{\infty} \Phi_m(\alpha)\, e^{i(m\theta+\omega t)}$$

$$F(\mu,J,\alpha,\beta) = F_o(\mu,J,\alpha) + \sum_{m=-\infty}^{\infty} F_m(\mu,J,\alpha)\, e^{i(m\theta+\omega t)}, \tag{30}$$

and linearizing equation (29), we have:

$$\oint \nabla \cdot \left[\overleftrightarrow{\epsilon} \cdot \nabla \Phi_m\right] \frac{ds}{B} = -8\pi \Sigma e \quad d\mu dJF_m \; . \tag{31}$$

To determine F_m, the linearized form of the Fourier-analyzed Liouville equation of equation (14) is needed:

$$i(\omega_m + \frac{cm}{e} \frac{\partial K_o}{\partial \alpha})\, F_m = i\frac{c}{e}m\, K_m \frac{\partial F_o}{\partial \alpha} \,, \tag{32}$$

where from equations (4) and (11) with $\psi = 0$, the K_o and K_m of

$$K(\alpha,\beta,J,\mu,t) = K_o(\alpha,J,\mu) + \sum_{m=-\infty}^{\infty} K_m(\alpha,J,\mu) e^{i(m\theta+\omega_m t)} \tag{33}$$

are to be determined from

$$J = \oint \frac{ds}{c} \left[(K-e\Phi)^2 - m_o^2 c^4 - 2\mu B m_o c^2\right]^{\frac{1}{2}}. \tag{34}$$

Since Φ is zero in equilibrium, and B is undisturbed in the perturbation, the differential of equation (34) gives for K_m,

$$\oint \frac{ds}{c} \frac{K_o(K_m - e\Phi_m)}{\left[K_o^2 - m_o^2 c^4 - 2\mu B m_o c^2\right]^{\frac{1}{2}}} = 0,$$

i.e.,

$$K_m(\alpha, J, \mu) = e\Phi_m(\alpha) \ . \tag{35}$$

Similarly, in the equilibrium state, equation (34) reads

$$J = \oint \frac{ds}{c} \left[K_o^2 - m_o^2 c^4 - 2\mu B m_o c^2\right]^{\frac{1}{2}} , \tag{36}$$

and $\frac{\partial K_o}{\partial \alpha}$ (α, μ, J) may be determined from this by differentiating with respect to α holding μ and J fixed, i.e.,

$$\oint \frac{ds}{c} \frac{1}{\left[K_o^2 - m_o^2 c^4 - 2\mu B m_o^2 c^4\right]^{\frac{1}{2}}} \left(K_o \frac{\partial K_o}{\partial \alpha} - \mu m_o c^2 \frac{\partial B}{\partial \alpha}\right) +$$

$$\oint \frac{\partial}{\partial \alpha}(ds) \frac{1}{c} \left[K_o^2 - m_o^2 c^4 - 2\mu B m_o^2 c^4\right]^{\frac{1}{2}} = 0. \tag{37}$$

Since $\vec{i}_\alpha$ is perpendicular to $\vec{B}$, and since $\nabla \times \vec{B} \approx 0$ for this case in which the magnetic energy density is much larger than the partial energy density, we have

$$B(s,\alpha,\beta)ds(s,\alpha,\beta) = B(s,\alpha + \partial\alpha,\beta)ds(s,\alpha + \partial\alpha,\beta)$$

from which it follows that

$$\frac{\partial}{\partial \alpha}(ds) = - \frac{1}{B} \frac{\partial B}{\partial \alpha} ds. \tag{38}$$

Equations (37) and (38) give

$$\frac{\partial K_o}{\partial \alpha} = \left[\oint \frac{K_o ds}{c\left[K_o^{\,2} - m_o^{\,2}c^4 - 2\mu B m_o c^2\right]^{\frac{1}{2}}} \right]^{-1} \oint ds \, \frac{1}{B}\frac{\partial B}{\partial \alpha}$$

$$\times \frac{\left[K_o^{\,2} - m_o^{\,2}c^4 - \mu B m_o c^2\right]}{c\left[K_o^{\,2} - m_o^{\,2}c^4 - 2\mu B m_o c^2\right]^{\frac{1}{2}}} \quad . \tag{39}$$

This may be expressed in more familiar terms by recognizing that the parallel and gyration velocities in the equilibrium state are simply

$$V_{\parallel} = \frac{c}{K_o}\left[K_o^{\,2} - m_o^{\,2}c^4 - 2\mu B m_o c^2\right]^{\frac{1}{2}} \tag{40}$$

$$V_{\perp} = \left(\frac{2\mu B}{m_o \gamma^2}\right)^{\frac{1}{2}} \tag{41}$$

since $K_o = m_o \gamma c^2$ and $\vec{p} = m_o \gamma \vec{V}$; in terms of equations (40)and (41), equation (39) may be rewritten

$$\frac{\partial K_o}{\partial \alpha} = \left(\oint \frac{ds}{V_{\parallel}} \right)^{-1} \oint \frac{ds}{V_{\parallel}} \left(\frac{1}{B}\frac{\partial B}{\partial \alpha}\right) \left[K_o^{\,2} - m_o^{\,2}c^4 - \mu B m_o c^2\right] \frac{1}{K_o}$$

$$= \left(\oint \frac{ds}{V_{\parallel}} \right)^{-1} \oint \frac{ds}{V_{\parallel}} \left(\frac{1}{B}\frac{\partial B}{\partial \alpha}\right) m_o \gamma \left(V_{\parallel}^{\,2} + \frac{V_{\perp}^{\,2}}{2}\right) \tag{39a}$$

Inserting equations (35) and (39a) into equation (32), we have

$$F_m = \frac{cm\Phi_m \frac{\partial F_0}{\partial \alpha}}{\omega_m + \frac{cm}{e}\left(\oint \frac{ds}{V_\parallel}\right)^{-1} \oint \frac{ds}{V_\parallel}\left(\frac{1}{B}\frac{\partial B}{\partial \alpha}\right) m_0 \gamma \left(V_\parallel^{\ 2} + \frac{V_\perp^{\ 2}}{2}\right)} \quad . \tag{42}$$

To proceed further with equation (31), the form of the dielectric tensor must be specified. The belt parameters suggest that at the frequencies of interest the polarization and resistive currents are provided primarily by very low energy particles (Cf. Section III). Fejer (1960) has discussed, in connection with the ionosphere, the dielectric tensor which obtains from a zero temperature three fluid (electrons, ions and neutrals) plasma, and we shall use his results for the polarization and resistive currents $\vec{j}_{pol} + \vec{j}_{res}$:

$$\vec{j}_{pol} + \vec{j}_{res} = \frac{\sigma_0(\vec{E}\cdot\vec{B})\vec{B}}{B^2} + \sigma_1\left[\vec{E} - \frac{(\vec{E}\cdot\vec{B})\vec{B}}{B^2}\right] + \sigma_2 \frac{\vec{E}\times\vec{B}}{B} \tag{43}$$

where

$$\sigma_0 = \frac{ne^2}{m_e}\left(\omega_i + \frac{m_e}{m_i}\omega_e\right)\left[i\omega_i\omega_e + \nu^*\left(\omega_i + \frac{m}{M}\omega_e\right)\right]^{-1}$$

$$\sigma_1 = \frac{ne^2}{m_e}\left(\omega_i + \frac{m_e}{m_i}\omega_e\right)\left(\omega_i - \frac{m}{M}\omega_e\right)\frac{\Omega_e}{D} \tag{44}$$

$$\sigma_2 = \frac{ne^2}{m_e}\left(\omega_i + \frac{m_e}{m_i}\omega_e\right)\frac{1}{D}\left[i(\omega_i\omega_e + \Omega_i\Omega_e) + \nu^*\left(\omega_i + \frac{m}{M}\omega_e\right)\right] .$$

Here the subscripts (i, e, 0) refer to (ions, electrons and neutrals), and

$$\omega_j = \omega \left[1 + \frac{\nu_j}{i\omega + \frac{\rho_i}{\rho_o}\nu_i + \frac{\rho_e}{\rho_o}\nu_e} \right]$$

$$\nu^* = \nu + \frac{\rho_i}{\rho_o}\nu_i\nu_e \frac{1}{i\omega + \frac{\rho_i}{\rho_o}\nu_i + \frac{\rho_e}{\rho_o}\nu_e} \tag{45}$$

$$D = (\omega_i^{\,2} - \Omega_i^{\,2})(\omega_e^{\,2} - \Omega_e^{\,2}) - 2i\nu^*(\omega_i + \frac{m_e}{m_i}\omega_e)(\omega_i\omega_e - \Omega_i\Omega_e) - \nu^{*2}(\omega_i + \frac{m_e}{m_i}\omega_e)^2$$

with ν denoting the collision frequency between ions and electrons, ν_i denoting the collision frequency of ions with neutrals, ν_e the collision frequency of electrons with neutrals, ρ the mass density, n the electron number density and Ω_i and Ω_e the ion and electron gyrofrequencies.

For frequencies of the order of the azimuthal drift frequencies of the energetic protons in the belt, it can be shown that the motion of the neutrals may be ignored ($\omega >> \rho_i\nu_i/\rho_o$), and that in the ionosphere ω and ν may be ignored compared to ν_j [Francis and Karplus (1960)]. In the ionosphere, then, we shall approximate σ_0, σ_1, and σ_2 by

$$\sigma_0 \approx ne^2 \sum \frac{1}{m_j\nu_j}$$

$$\sigma_1 \approx ne^2 \sum \frac{\nu_j}{m_j\left[\nu_j^{\,2} + \Omega_j^{\,2}\right]} \tag{44a}$$

$$\sigma_2 \approx ne^2 \Sigma \frac{\Omega_j}{m_j \left[\nu_j^2 + \Omega_j^2\right]},$$

In the belt itself, with $\Omega_j >> \omega >> \nu >> \nu_j$ we have

$$\sigma_0 \approx \frac{ne^2}{m_e i\omega}$$

$$\sigma_1 \approx \frac{i\omega}{i\omega \frac{m_e}{ne^2} + \frac{B^2}{\rho_i + \rho_e}} \quad , \; \sigma_2 \approx 0. \tag{44b}$$

With $\overleftrightarrow{\epsilon} = 1 + \frac{4\pi c^2}{i\omega} \overleftrightarrow{\sigma}$, insertion of equations (42) and (43) in equation (31) then yields the normal mode equation

$$\oint ds\, B \frac{\partial}{\partial\alpha}\left[r^2 \left(1 + \frac{4\pi\sigma_1}{i\omega}\right) \frac{\partial\Phi_m}{\partial\alpha}\right] - \left[m^2 \oint \frac{ds}{Br^2} \left(1 + \frac{4\pi\sigma_1}{i\omega}\right)\right.$$

$$\left. + \oint ds\, B \frac{\partial}{\partial d} \left(\frac{4\pi\sigma_2 m}{B\omega}\right) \right] \Phi_m = -8\pi\Phi_m \Sigma \iint d\mu\, dJ$$

$$\times \frac{ecm \frac{\partial F_0}{\partial\alpha}}{\omega + \frac{cm}{e}\left(\oint \frac{ds}{V_\parallel}\right)^{-1} \oint \frac{ds}{V_\parallel} \left(\frac{1}{B}\frac{\partial B}{\partial\alpha}\right) m_o\gamma \left(V_\parallel^{\,2} + \frac{V_\perp^{\,2}}{2}\right)} \tag{46}$$

$$= -\mathcal{M}(r_E)\Phi_m$$

It is useful to express the integral on the right-hand side of equation (46) in terms of a more familiar distribution function. In the following, we shall restrict our attention to a dipole field of magnetic moment $\vec{M}$:

$$\chi = \frac{M\cos\Theta}{r^2} \quad , \tag{47}$$

denoting the polar angle by θ and the distances from $\vec{M}$ by r. Hamlin, Karplus, Vik and Watson (1961) have shown that the denominator $\mathscr{D}$ in the integral on the right-hand side of equation (46) may be approximated by the simple expression

$$D \approx \omega - \frac{m3cr_E}{eM} \left(\frac{E^2 - m_o^2c^4}{E}\right) (0.35 + 0.15 \sin\alpha_E') \ , \tag{48}$$

where E is the relativistic energy of the particles and α_E' and r_E are the helix angle and distance of the particle from $\vec{M}$ when it crosses the equator $\theta = \pi/2$. A convenient equilibrium distribution function to introduce is then $f_o(r_E, \frac{\pi}{2}; \alpha_E', E)\, dVd\alpha_E'dE$, this denoting the number of particles in the volume dV, equatorial helix angle interval $d\alpha_E'$ and energy interval dE at the equatorial distance r_E, helix angle α_E' and energy E. Denoting by $f_o(r,\theta; \alpha', E)\, dVd\alpha'dE$ the number of particles in the interval $dVd\alpha'dE$ at a general point in the field r,θ (ignoring the azimuthal coordinate Φ since the equilibrium distribution is assumed independent of Φ), Liouville's theorem and the constancy of the magnetic moment then g·

$$f_o(r,\theta, E, \alpha') = \frac{(1 + 3\cos^2\theta)^{1/4}}{\sin^3\theta} \, f_o\left(r_E = \frac{r}{\sin^2\theta}, \frac{\pi}{2}, E, \right.$$

$$\left. \alpha_E' = \sin^{-1}\left\{\frac{\sin^3\theta \sin\alpha'}{(1+3\cos^2\theta)^{1/4}}\right\}\right) \quad . \tag{49}$$

The differential volume of a tube of flux $d\alpha d\beta$ contained in the differential angle $d\Theta$ is

$$dV = \frac{r_E^4}{M} \; d\alpha d\beta \, \sin^7 \Theta d\Theta, \tag{50}$$

where r_E is the distance of the tube from $\vec{M}$ at the equator. Thus, since an interval $d\mu dJ$ is uniquely determined by an interval $d\alpha'_E dE$ at a given r_E, we have:

$$f(\mu, J, \alpha, \beta) d\mu dJ = \int d\Theta \, \frac{r_E^4}{M} \sin^7\Theta \; f_o(r, \Theta, E, \alpha') \, \frac{\partial \alpha'_E}{\partial E} \, d\alpha'_E dE \tag{51}$$

where $f_o(r, \Theta, E, \alpha)$ is given by equation (49) and where because $\frac{\sin^2\alpha'}{B}$ is invariant,

$$\frac{d\alpha'}{d\alpha'_E} = \frac{(1 + 3\cos^2\Theta)^{1/4}}{\sin^3\Theta} \; \frac{\cos \alpha'_E}{\left[1 - \frac{(1 + 3\cos^2\Theta)^{\frac{1}{2}}}{\sin^6\Theta} \; \sin^2\alpha'_E\right]^{\frac{1}{2}}} . \tag{52}$$

A particularly simple choice for $f_o(r_E, \frac{\pi}{2}; \alpha'_E, E)$ is

$$f_o(r_E, \tfrac{\pi}{2}; \alpha'_E, E) = N(r_E, E) \sin^p \alpha'_E, \tag{53}$$

for then

$$f_o(r, \Theta, \alpha', E) = N\left(r_E = \frac{r}{\sin^2\Theta}, E\right) \frac{\sin^{3(p-1)}\Theta \; \sin^p \alpha'}{(1+3\cos^2\Theta)^{\frac{p-1}{4}}} , \tag{54}$$

and

$$F(\mu,J,\alpha,\beta)\ d\mu dJ = \frac{2r_E^4}{M} N(r_E,E) \sin^p\alpha_E' \cos\alpha_E'$$

$$\int_{\theta_c}^{\pi/2} \frac{d\theta \sin\theta\ (1 + 3\cos^2\theta)^{\frac{1}{2}}}{\left(1 - \frac{(1+3\cos^2\theta)^{\frac{1}{2}}}{\sin^6\theta} \sin^2\alpha_E'\right)^{\frac{1}{2}}} \quad d\alpha'_E dE \tag{55}$$

where

$$\frac{\sin^6\theta_c}{(1 + 3\cos^2\theta_c)^{\frac{1}{2}}} = \sin^2\alpha'_E\ . \tag{56}$$

When equation (53) applies, then, we have with $d\mu dJ$ proportional to r_E^4 ,

$$\iint d\mu dJ \quad \frac{\frac{\partial F_o}{\partial\alpha}}{\omega + \frac{cm}{e}\left(\oint \frac{ds}{V_\parallel}\right)^{-1} \oint \frac{ds}{V_\parallel} \left(\frac{1}{B}\frac{\partial B}{\partial\alpha}\right) m_o\gamma\ \left(V_\parallel^{\ 2} + \frac{V_\perp^2}{2}\right)} =$$

$$\frac{2r_E}{M^2} \int dE \frac{\partial}{\partial r_E} \quad N(r_E,E) \int d\alpha'_E \sin^p \alpha'_E \cos\alpha'_E \quad \frac{\mathcal{J}(\alpha'_E)}{\mathcal{D}(\alpha'_E)} \tag{57}$$

where

$$\mathcal{J}(\alpha'_E) = \int_{\theta_c}^{\pi/2} d\theta \quad \frac{\sin\theta\ (1 + 3\cos^2\theta)^{\frac{1}{2}}}{\left(1 - \frac{(1+3\cos^2\theta)^{\frac{1}{2}}}{\sin^6\theta} \quad \sin^2\alpha_E'\right)^{\frac{1}{2}}} \tag{58}$$

and $\mathcal{B}$ defined by Eq. (48).

iii Discussion

In analyzing equations (46) and (57) further, it is well to keep in mind some specific features of the Van Allen belt and the ionosphere. In the radiation belt, whistler measurements and particle detection experiments indicate the presence of a plasma the bulk of which is of low energy and of which only a small fraction comprises the high energy (radiation) component. Thus, referring to O'Brien (1964) for a summary of the high energy data, the energetic proton spectrum is described* by the empirical relation of McIlwain and Pizella (1963)

$$j(E_p)\, dE = \text{constant}\ e^{-E_p/kT}\, dE,$$

$$kT = (306 \pm 28) L^{-(5.2 \pm 0.2)}\ \text{Mev for}\ 1.2 \leq L \lesssim B,$$

where $j(E_p)$ denotes the intensity of protons of energy E_p and L is the magnetic shell parameter introduced by McIlwain (1961). A typical flux of protons in the 1.1-14 Mev range at $L \sim 2$ might be $3 \times 10^6\ \text{sec}^{-1}\ \text{cm}^{-2}\ \text{ster}^{-1}$ [McIlwain, et al (1964)]. The energetic electron spectra is summarized by O'Brien in the form of two graphs, one for the inner and the other for the heart of the outer zone: the inner zone graph indicating a relatively flat flux spectrum extending

*. Although recent measurements [McIlwain et.al. (1964)] indicate that this relation may not always apply.

from ~ 50 kev up to ~ 500-600 kev with a flux of $\sim 10^4$ electrons/kev cm^2 ster. sec, and the outer zone graph showing a typical intensity of $\sim 10^5$ electrons/cm^2 sec kev dropping off quite abruptly between 10^3 and 10^4 kev. Referring to Liemohn and Scarf (1964) for a summary of the low energy electron data obtained from whistlers, the low energy electron density $N(r,\theta)$ is described by functions of the form

$$N(r,\theta) = r^{-3}\, G(r,\theta),$$

where r and θ denote radial distance and geomagnetic latitude, respectively, and $G(r,\theta)$ is a smooth function of r and θ in the range $3 < r/R_E < 5$ (R_E is the earth's radius). One such function for equatorial densities is

$$N = 14100\ (R_E/r)^3 \text{ electrons/cm}^2,$$

(as given in their figure 6). Liemohn and Scarf attribute the whistler damping to non-thermal electrons of energies in the range 0.2-2 kev in the tail of the energy distribution. Although they conclude that a temperature for the distribution cannot be estimated from whistler data, it would appear that the bulk of the electrons have much lower energy; e.g., an earlier analysis (Scarf, 1962, Liemohn and Scarf, 1962) showed a temperature of $2 \times 10^{5\,\circ}$K. From the foregoing, then, we are led to consider a radiation belt plasma in which the number of low energy electrons and ions is much larger than the number of high energy electrons and ions, but in which the total energy associated with the high energy component is much larger than that of the low energy component.

For ionospheric data, reference may be made to the review article by Van Zandt and Knecht (1964) and to the article on hydromagnetic waves in the ionosphere by Francis and Karplus (1960). Here we find that σ_1 and σ_2 are sharply peaked functions of r whose widths and heights are proportional and inversely proportional to Ω_i, respectively. If all terms in σ_1 and σ_2 except Ω_i are assumed independent of r_E, an approximation which neglects the dependence of the ionosphere upon latitude--we have already neglected longitudinal variations by the assumption of cylindrical symmetry, and if it is further noted that the positions of the peaks are a slow function of r_E, we can make the approximation that σ_1 and σ_2 are independent of r_E and write

$$\sigma_1 \approx \frac{\omega_p^2}{2\Omega_i} \Delta r\, \delta(r-r_1) = 4\pi N_1 \delta(r-r_1)$$

$$\approx \frac{\omega_{pi}^2}{\Omega_i} \Delta r\, \delta(r-r_2) = 4\pi N_2 \delta(r-r_2)$$

$$\omega_{pi}^2 = \frac{4\pi e^2 n}{M_i} \tag{59}$$

where, from Francis and Karplus (1960) and Van Zandt and Knecht (1964), $\Delta r \approx 5 \times 10^6$ cm, $r_1 \approx R_E + 140$ km, $r_2 \approx R_E + 120$ km, at which heights the ionospheric electron density $n \approx 2 \times 10^5 (5 \times 10^3)$ cm^{-3} during the day (night). In the following, we shall set r_1, $r_2 = R_E$. As representative parameters, we have at 120 km above the earth's surface, a neutral density of 6×10^{11} cm^{-3}, an electron (ion) gyrofrequency of 8×10^6 (2×10^2) and a collision frequency with neutrals of 8×10^3 (3×10^2) sec^{-1} for electrons (ions). These frequencies are considerably larger than the azimuthal drift frequencies of the energetic radiation belt particles. The distributions employed are

$$N_H^j(r_E,E) = \frac{3}{2} \frac{n_H}{\sqrt{\pi E_H^j}} \frac{e^{-E/kT_H^j}}{\sqrt{E}}$$

$$kT_H^j = E_H^j \left(\frac{R_E}{r_E}\right)^5 \tag{60}$$

$$p = 3$$

and

$$N_L(r_E,E) = \frac{2}{\pi^{3/2}} \frac{n_L}{(kT_L)^{1/2}} e^{-E/kT_L} \frac{1}{\sqrt{E}} \left(\frac{R_E}{r_E}\right)^3 \tag{61}$$

$$p = 1$$

wherein the latter form occurs since both electrons and ions are Maxwellized due to the large number of collisions as far as containment is concerned. Also note that the average energy of the ions is considerably greater than the average energy of the electrons for the high energy component. Next, the angular integrations in Eq. (57) are performed. To do this it is convenient to interchange the order of the integration over Θ and α_E'. Further, the non-relativistic approximation is utilized to evaluate $\mathcal{S}(\alpha_E')$ in addition we replace the α_E' dependence in this latter expression by its average value (a valid procedure since .35 dominates .15 sin α_E', i.e., we write $0.35 + 0.15 \sin \alpha_E' \approx 1/2$. We then obtain (see Eqs. 46 and 57)

$$\mathcal{M}(r_E) = \sum_{j,p=1,3} \frac{16\pi e_j mc}{M^2}\, 2r_E^2 \int dE\, \frac{\partial N_p^j}{\partial r_E}$$

$$\times \frac{1}{\omega - \frac{3mEr_E}{e_j M} - i\epsilon} \int_0^{\pi/2} \frac{(\sin\theta)^{4+p}}{(1+3\cos^2\theta)^{\frac{p-1}{4}}} \begin{Bmatrix} \frac{2}{3}, & p = 3 \\ 1, & p = 1 \end{Bmatrix}$$

or

$$\mathcal{M}(r_E) = \sum_j \frac{32\pi e_j mc r_E^6}{M^2} \frac{\partial}{\partial r_E} \int \frac{dE}{\sqrt{E}}$$

$$\left\{ \frac{}{\pi^{3/2}} \frac{n_L}{(kT_L)^{1/2}} e^{-E/kT_L} \left(\frac{R_E}{r_E}\right)^3 + \frac{.34 n_H}{\sqrt{\pi E_H^j}} e^{-E/kT_H^j} \right\} \times \frac{1}{\omega - \frac{3mEr_E c}{e_j M} - i\epsilon} \,. \tag{62}$$

The sign of ϵ is chosen in the standard Landau sense. Now if we define the dimensionless quantities

$$\xi = \xi_L^p = \frac{\omega e M}{3mkT_L c r_E} = -\xi_L^e$$

$$\xi_H^p = \frac{T_L}{T_H^p}\xi, \quad \xi_H^e = -\frac{T_L}{T_H^e}\xi$$

we obtain with $E/kT = x$

$$\mathcal{M}(r_E) = -\sum \frac{32\pi e^2 r_E^4}{3M} \int \frac{dx}{\sqrt{x}} e^{-x} \left\{ \frac{2.7 n_L}{\pi^{3/2}\, kT_L} \left(\frac{R_E}{r_E}\right)^3 \frac{1}{\xi_L^j - x} + \frac{1.7 n_H}{\sqrt{\pi}\, kT_H^j} \left(\frac{R_E}{r_E}\right)^{5/2} \frac{x}{\xi_H^j - x} \right\}. \tag{63}$$

We next turn to an analysis of stability. Since the time dependence is proportional to e^{iwt} it need only be shown that there are no roots to Eq. (46) in the l.h.p. (lower half ω-plane). To this end, we multiply this equation by $\omega\, \Phi_m^*$ and integrate over α to obtain

$$4\pi\int d\alpha\oint ds\; B\left\{-\lambda^2\left|\frac{d\Phi_m}{\partial\alpha}\right|^2\frac{\sigma_1}{i}-\frac{m^2\sigma_1}{B^2\lambda^2 i}|\Phi_m|^2-\left(\frac{\partial}{\partial\alpha}\frac{m\sigma_1}{B}\right)|\Phi_m|^2\right\} \tag{64}$$

$$+\,\omega\int d\alpha\,\mathcal{M}(r_e)\,|\Phi_m|^2\equiv Q(\omega)=0$$

wherein we only keep terms which contain components of the conductivity tensor in the integrals over the lines since these are the dominant terms. Also it has been assumed that at the end points Φ_m or $\frac{\partial\Phi_m}{\partial\alpha}$ vanish. Now the number of roots in the l.h.p. which satisfy Eq. (64) will be equal to the number of times $Q(\omega)$ encircles the origin as ω takes on values which bound the l.h.p. Assuming that Φ_m is a bounded function of ω we see that as $\omega\to\infty$ the last term in Eq. (64) vanishes and the imaginary part of Q is positive. In addition, as is easily seen from Eq. (63), in the $\omega = 0$ limit the last term on the l.h.s. of Eq. (64) vanishes and again Q is in the upper half plane. This latter result is independent of the detailed form of the distribution function in the radiation belt depending only upon the necessity that the density be finite, viz. the integral $\int f_0\frac{dE}{\sqrt{E}}$ does not diverge at $E = 0$. Finally, we are able to extract the sufficient, but not necessary, condition for stability

$$m^2\oint\frac{4\pi\sigma_1}{\lambda^2 B}\,ds>\omega\,\mathrm{Im}\left(-\mathcal{M}\right) \tag{65}$$

for all r_E and ω real. This is nothing but the condition that Im Q never vanishes on the real ω axis for all r_E. If Eq. (60) is incorporated we have, since the resonant frequencies are so well separated,

$$\frac{1}{4\sqrt{\pi}}\frac{\Delta r}{r_E}\left[\frac{m}{(1-R_E/r_E)(4-3R_E/r_E)}\right]>\frac{n_L}{n}\sqrt{\pi}\left|\xi_2^j\right|e^{-\left|\xi_2^j\right|},\sqrt{\pi}\left|\xi_H^j\right|^{3/2}\frac{n_H}{n}e^{-\left|\xi_H^j\right|}\left(\frac{r_E}{R_E}\right)$$

or

$$\frac{1}{4\sqrt{\pi}} \; \frac{\Delta r}{r_E} \frac{m}{\left[(1-R_E/r_E)(4-3\ R_E/r_E)\right]^{\frac{1}{2}}} > \frac{n_L}{n}\left(\frac{\pi}{2e}\right)^{\frac{1}{2}},\ \frac{n_H}{n}\left(\frac{3\pi}{4e}\right)^{\frac{1}{2}} \left(\frac{r_E}{R_E}\right)^{\frac{3}{2}} \tag{66}$$

after common factors have been removed.

Consider first the contribution from the high energy species; we have for the parameters listed in Eq. (60) with $n_H = 1$

$$m\left(\frac{R_E}{r_E}\right)^{3/2} \frac{1}{\left[(1-R_E/r_E)(4-3\ R_E/r_E)\right]^{\frac{1}{2}}} > 5 \times 10^{-3} \quad \text{day}$$

$$> 0.2 \quad \text{night}$$

Hence we see that the night time values of the parameters in the ionosphere are on the border, for high lines of violating our overly pessimistic stability criteria. Needless to say, the fact that for $n_L \approx 10^4$ the contribution from the low energy species violates our stability criteria.

To obtain further information about the presence and magnitude of the growth times when the stability criteria is violated we resort to a local treatment of Eq. (46), an approximation that is well justified for high m modes.

If we now define the quantities

$$g = \oint ds\ B\, r^2 \left(1 + \frac{4\pi}{i\omega}\sigma_1\right)$$

$$g_\alpha = \oint ds\ B \frac{\partial}{\partial\alpha} r^2 \left(1 + \frac{4\pi}{i\omega}\sigma_1\right)$$

$$h = \oint \frac{ds}{B r^2} \left(1 + \frac{4\pi}{i\omega}\sigma_1\right)$$

$$k = \frac{4\pi}{\omega} \oint ds\ B \frac{\partial}{\partial\alpha} \frac{\sigma_2}{B}$$

we can rewrite Eq. (46) in the form

$$\left\{ g \frac{r_E^2}{M^2} \frac{\partial^2}{\partial r_E^2} + \left[mk - m^2 h - \frac{g_\alpha^2}{4g} - \frac{g}{2} \frac{\partial}{\partial\alpha} \frac{g_\alpha}{g} + \mathcal{M}(r_E) \right] \right\} \frac{r_E \phi}{\Lambda} = 0 \qquad (46a)$$

where

$$\frac{1}{\Lambda} \frac{\partial\Lambda}{\partial\alpha} = -\frac{g_\alpha}{2g}$$

and

$$\frac{\partial}{\partial\alpha} = \frac{M}{r^4 B^2} \frac{\partial}{\partial r} + \frac{r_E^2}{M} \frac{\partial}{\partial r_E} .$$

Incorporating Eq. (60) we now have

$$\left[\varepsilon \frac{r_E^2}{M^2} \frac{\partial^2}{\partial r_E^2} + G + \mathcal{M}(r_E) \right] \phi \frac{r_E}{\Lambda} = 0, \qquad (67)$$

where

$$G = -m^2 \frac{4r_E^2}{M} \sqrt{1-R_E/r_E} - \frac{8\pi N_1 r_E}{i\omega M} \frac{1}{\sqrt{1-R_E/r_E}} \left[m^2 + \frac{5}{16} \frac{1}{(1-R_E/r_E)^2} \right.$$

$$\left. - \frac{3}{16} \frac{1}{1-R_E/r_E} + \frac{9}{16} \frac{R_E}{r_E} - \frac{1}{8} - \frac{4}{4-3R_E/r_E} \right] + \frac{4\pi N_2 m R_E}{M\omega}$$

$$\times \left[(1-R_E/r_E)(4-3R_E/r_E) \right]^{-3/2}, \tag{68a}$$

and

$$g = M\sqrt{1-R_E/r_E}\; (2-R_E/r_E) + \frac{8\pi N_1 M}{i\omega r_E} \frac{4-3R_E/r_E}{\sqrt{1-R_E/r_E}} \,. \tag{68b}$$

In the above equations we have replaced $r_{1,2}$ by R_E since relative to the values of r_E in the Van Allen belt the errors so encountered are insignificant. Note that the minimum value of the term is the first square bracket in m^2-1.

With the restriction to a local treatment the defining equation for the eigenvalues is

$$D\ (r_E,m,\omega) \equiv \omega\ (G+\mathcal{M}) = 0 \tag{69}$$

The above equation has been multiplied by ω for convenience. Also in Eq. (69) we keep only those terms in G which come from the conductivity tensor in the ionosphere, since in the average over lines of force those terms dominate. Here too in both limits $\omega \to \infty$ and $\omega \to 0$ D is in the second quadrant and once again we have essentially the sufficient stability criteria of Eq. (65). In the event that this condition be violated (the solutions to the above equation must occur in pairs), a necessary and sufficient condition for instability is that Re $D(m,\omega)$ be of opposite signs for any pair of solutions to the equality.

In general, $D(m,\omega)$ is proportional to

$$\frac{1}{\sqrt{\pi}}\int dx\, e^{-x}\sum_j\left\{\frac{1}{\sqrt{x}}\,\frac{\xi_L^j}{x-\xi_L^j}+2\left(\frac{r_E}{R_E}\right)^{\frac{1}{2}}\frac{n_H}{n_L}\sqrt{x}\,\frac{\xi_H^j}{x-\xi_H^j}\right\}\frac{e_j}{|e|}$$

$$+\frac{n}{n_L}\,\frac{\Delta r}{r_E}\,\frac{1}{4\sqrt{\pi}}\left[(4-3\,R_E/r_E)(1-R_E/r_E)\right]^{-\frac{1}{2}}\left\{\frac{i(m^2-1)}{m}+\frac{R_E}{r_E}\,\frac{1}{1-R_E/r_E}\,\frac{1}{(4-3R_E/r_E)^{3/2}}\right\} \tag{70}$$

In arriving at Eq. (70) we have utilized Eqs. (60) and (68). Note that for convenience we have replaced $\mathrm{Im}(-G)$ by its minimum value, an approximation that will underestimate the growth rate. It is of interest to record the integral in Eq. (70) in terms of the dispersion integral of (Fried and Conte)

$$\sum_j\frac{e_j}{|e|}\left\{2\,\frac{n_H}{n_L}\,\xi_H^j\left[1+\sqrt{\xi_H^j}\,Z\,(\sqrt{\xi_H^j})\right]+\sqrt{\xi_L^j}\,Z\,(\sqrt{\xi_L^j})\right\} \tag{71}$$

At this point it is convenient to sketch, in Fig. 2, the behavior at the above expression, Eq. (71), for the following range of parameters

$$\frac{n_L}{n_H} \gg 1,\quad \frac{n_H T_H^p}{n_L T_L} > 1,\quad \frac{T_H^e}{T_H^p} < 1$$

If we assume the ratio of the temperatures of the high energy species is of the order of unity then the plot of the real part for $\xi \ll -1$ is analogous to the mirror image of the plot for $\xi \gg 1$. If the ratio is much greater than one then the sketch of the real part for $|\xi| \gg 1$ should be inverted through the origin.

As seen from the graph, for the present parameters, the condition for instability, namely, that the ReD be of opposite signs for the pair of frequencies at which Im D = 0 is easily satisfied by the low energy contribution and might possibly occur for the high energy contribution with the night time ionospheric parameters. Rather than analize our stability criteria in more detail for this latter case we directly solve for the growth. If we take the assymptotic form of the integrals we obtain for large r_E/R_E

$$\xi_H^p = \frac{2}{5}\left(\frac{r_E}{R_E}\right)^{3/2}\left\{-\frac{i}{m} + \frac{R_E}{r_E}\ \frac{1}{8m^2}\right\}$$

which is valid provided $\frac{r_E}{R_E} > 3m^{2/3}$. If in order to find the unstable root it is necessary to use the convergent expansion the growth time will be of no significance since it will be long compared to the drift time of the energetic particles. Thus it might be conjectured that the night time stability criteria determine the belt parameters for the energetic particles.

Returning now to the contribution from the low energy species we see that in general our criteria for instability is satisfied twice leading to two unstable roots. By taking the asymptotic and convergent approximation respectively to the integral in Eq. (70) we obtain

$$\mathrm{Im}\ \xi_L \approx -\frac{1}{m}\,10^2\,\frac{r_E}{R_E} \quad \text{day} \quad (\times\ 40 \text{ night}) \tag{72a}$$

and

$$\mathrm{Im}\ \xi_L \approx -\frac{10^{-4}}{\pi}\left(\frac{R_E}{r_E}\right)^2 m^2 \quad \text{day} \quad (\times\ \tfrac{1}{40} \text{ night}) \tag{72b}$$

In arriving at the above results we have neglected the real part of G compared to the imaginary part which is valid provided we treat the quantity $1 - R_E/r_E \approx 1$. Note that this assumption was implicit in our treatment of integrals over ionospheric parameters. Hence we have for the growth time, T_γ from Eq. (72a),

$$T_\gamma = T_D \, 10^{-2} \, \frac{R_E}{r_E} \, \frac{T_H^p}{T_L}$$

where T_D is the time it takes a high energy proton to drift around the earth. Consequently, the instabilities associated with the low energy species do not violate the third adiabatic invariant of the energetic particles.

In conclusion the following remarks are in order. The calculation is in the $\beta = 0$ local approximation. Our restriction to cylindrical symmetry, a condition violated by the asymmetry of the solar wind, restricts our attention to larger m-modes. In addition our assumption regarding the independence of ionospheric parameters upon polar angle is an approximation in the same view as that of the previous assumption. Our choice of separable dependence upon pitch angle and energy of the distribution function for the energetic particles is not completely consistent with the present data. Fortunately, the results do not appear to be sensitive to the particular form of the high energy distribution. Inasmuch as the data is still in a state of flux, it is obvious that a stability analysis based on the developed theory can be at best an order of magnitude result.

Keeping in mind the aforementioned restrictions our results can be summarized as follows: If we assume that most of the energy of the belt resides in the energetic protons, and if we look at disturbances occurring fast enough to violate the third adiabatic invariant of these energetic protons, then this dispersion formula essentially just balances resistive ionospheric currents against the $\vec{E} \times \vec{B}$ and ∇B driving currents of only the belt's energetic components. The higher the conductivity of the ionosphere, the harder it is to maintain the driving electric fields which could give an instability. A very high conductivity would prevent harmful interchanges (see Eq. 66). It appears that the electron density which exists in the ionosphere in the daytime is more than adequate to make the energetic component stable against interchanges. On the other hand, the nighttime ionospheric density is lower, and it appears that this lower nighttime ionospheric density might place the belt's energetic component close to marginal stability. This suggests that a simple electrostatic flute instability may play some role in limiting the energy content of the belt.

APPENDIX

In this appendix we demonstrate that the effect of a parallel electric field is negligible thus confirming the approximation $\frac{\partial \Phi_m}{\partial s} = 0$. First, we remark that Φ_m can only be dependent upon s in the ionosphere since in the belt proper the conductivity along the lines is extremely large. The second point to note is that although the spatial derivatives of the components of the conductivity tensor are inversely proportional to their widths, when averaged over a line of force there occurs a cancellation with the result that the derivatives are inversely proportional to the radial dimensions (see for example Eq. (68b)). Consequently, if Φ_m varies along the line in the region where the conductivity tensor is peaked, in general the cancellation will not occur and it would be expected that these terms contain the most significant effects of the presence of a parallel electric field. To estimate the effect we solve by iteration; if we write

$$\Phi_m(\alpha,s) = \Phi_m^0(\alpha) + \Phi_m^1(\alpha,s) \tag{A-1}$$

where Φ_m^1 is assumed small and Φ_m^0 is the solution to Eq. (46) in the absence of a parallel electric field. If Eq. (31) is recalled we have with Eq. (46) the equation for Φ_m^1

$$\frac{\partial}{\partial s} \sigma_0 \frac{\partial \Phi_m^1}{\partial s} = - B^2 \frac{\partial}{\partial_\alpha} \mathcal{R}^2 \sigma_1 \frac{\partial \Phi_m^0}{\partial \alpha} + mB^2 \left(\frac{\partial}{\partial_\alpha} i\sigma_2\right) \Phi_m^0 \tag{A-2}$$

where the r.h.s. of Eq. (A-2) contain the terms previously mentioned.

If we now write

$$\frac{\partial}{\partial_s} = \frac{\vec{B}}{B} \cdot \nabla = - 2 \left(\frac{1-r/r_E}{4-3r/r_E}\right)^{\frac{1}{2}} \frac{\partial}{\partial r} \tag{A-3}$$

with r and r_E independent variables we can, if we utilize

$$\frac{\partial}{\partial_\alpha} = \frac{M}{r^4 B^2}\frac{\partial}{\partial r} + \frac{r_E^2}{M}\frac{\partial}{\partial r_E} \tag{A-4}$$

write

$$\frac{\partial r}{\partial \alpha} = -\frac{r^2}{2M}\left(\frac{1}{1-r/r_E}\,\frac{1}{4-3r/r_E}\right)^{\frac{1}{2}}\frac{\partial r}{\partial s} \tag{A-5}$$

which when substituted into Eq. (A-2) yields after integrating once over s

$$\frac{\partial \Phi_m^1}{\partial s} = \frac{M}{2R_E^4}\left(\frac{4-3\;R_E/r_E}{1-R_E/r_E}\right)^{\frac{1}{2}}\left[R_E^2\,\frac{\sigma_1}{\sigma_0}\,\frac{\partial \Phi_m^0}{\partial_\alpha} - \frac{im\sigma_2}{B\sigma_0}\,\Phi_m^0\right] \tag{A-6}$$

To arrive at the above result we have utilized the following properties: $\frac{\partial \Phi_m^1}{\partial s} = 0$ in the belt, $\sigma_1, \sigma_2 \rightarrow 0$ in the belt (see discussion preceding Eq. (60)), in all terms which are gentle functions of r we have set $r_E = R_E$ an approximation consistent with the strongly peaked nature of σ_1 and σ_2. If Eq. (A-6) is now inserted back into the appropriate terms of Eq. (46) we obtain, after an integration by parts, in addition to the l.h.s. of Eq. (46) the terms

$$\frac{4\pi}{\omega}\,\frac{1}{1-R_E/r_E}\oint ds\left\{\frac{B\sigma_1^2}{i4\sigma_0}\,\frac{\partial \Phi_m}{\partial_\alpha} - \frac{m}{R_E^2}\,\frac{\sigma_1^2}{4\sigma_0}\,\Phi_m + \frac{m^2}{4R_E^2}\,\frac{\sigma_1\sigma_2}{i\sigma_0}\,\Phi_m - \frac{m\sigma_2^2}{4\sigma_0}\,B\,\frac{\partial \Phi_m}{\partial_\alpha}\right\} \tag{A-7}$$

If Eq. (46) is now compared with Eq. (A-7) it is immediately obvious that, provided $\frac{\sigma_1, \sigma_2}{\sigma_0}$ is small, the contribution from the parallel electric field is insignificant. For the parameters of the ionosphere $\frac{\sigma_2}{\sigma_0}$ at the peak of σ_2 is on the order of 10^{-2} while $\frac{\sigma_1}{\sigma_0}$ at the peak of σ_1 is on the order of 10^{-3}; hence the approximation that Φ_m be constant along a line of force is a valid one.

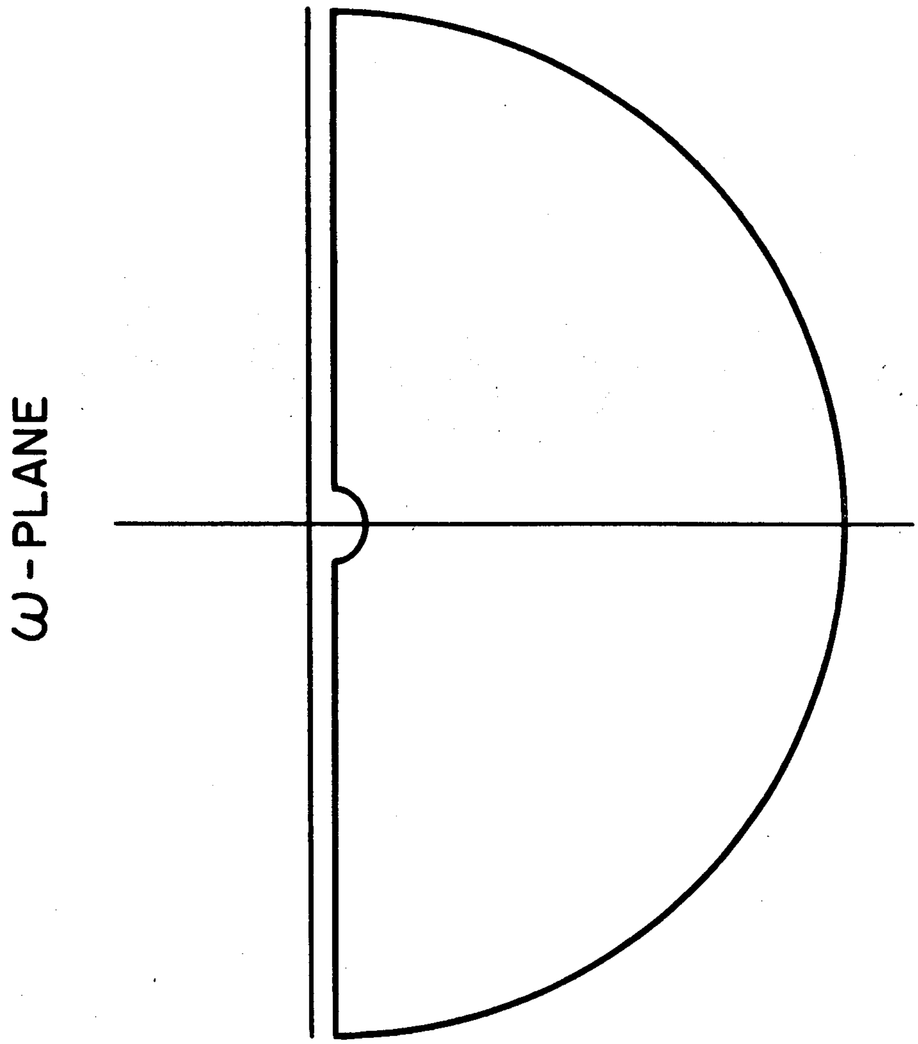

Fig. 1 -- Complex ω-Plane.

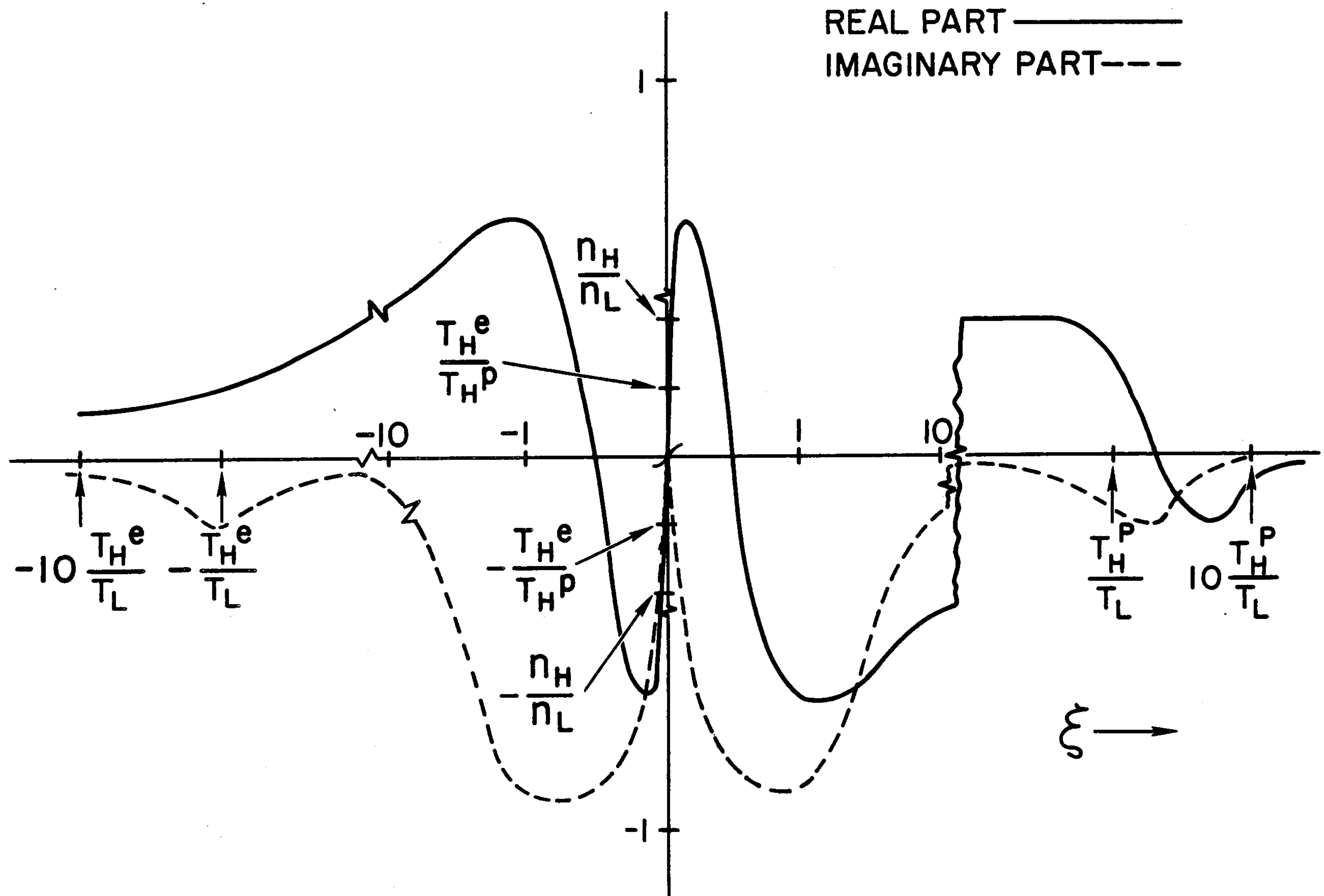

Fig. 2 -- Dispersion Integral

REFERENCES

Axford, W. I., and C. O. Hines, A unifying theory of high-latitude geophysical phenomena and geomagnetic storms, Can. J. Phys., 39, 1433-1463, 1961.

Fejer, J. A., Hydromagnetic wave propagation in the ionosphere, J. Atmospheric and Terrest. Phys., 18, 135-146, 1960.

Francis, W. E., and R. Karplus, Hydromagnetic waves in the ionosphere, J. Geophys. Res., 65, 3593-3600, 1960.

Gold, T., Motions in the magnetosphere of the earth, J. Geophys. Res., 64, 1219-1224, September, 1959.

Gold, T., Interchange and rotation of the earth field lines, J. Phys. Soc. Japan, 17, Suppl. A-1, 187-188, January, 1962.

Liemohn, H. B., and F. L. Scarf, Exospheric electron temperatures from nose-whistler attenuation, J. Geophys. Res., 67, 1785, 1962.

Liemohn, H. B., and F. L. Scarf, Whistler determination of electron energy and density distributions in the magnetosphere, J. Geophys. Res., 69, 883-904, 1964.

McIlwain, C. E., Coordinates for mapping the distribution of magnetically trapped particles, J. Geophys. Res., 66, 3681-3691, 1961.

McIlwain, C. E., R. W. Fillius, J. Valerio, and A. Davé, Relay I Trapped Radiation Measurements, UCSD preprint, March, 1964.

McIlwain, C. E., and G. Pizzella, On the energy spectrum of protons trapped in the earth's inner Van Allen zone, J. Geophys. Res., 68, 1811-1823, 1963.

Northrup, T. G., and E. Teller, Stability of the adiabatic motion of charged particles in the earth's field, Phys. Rev., 117, 215-225, 1960.

O'Brien, B. J., The trapped radiation zones, Chp. 14 in Space Physics, ed. D. P. LeGalley and A. Rosen, John Wiley and Sons, Inc., New York, 1964.

Rosenbluth, M. N., N. A. Krall, N. Rostoker, Finite Larmor radius stabilization of "weakly" unstable confined plasmas, Nuclear Fusion, 1962 Supplement Part 1, 143-150, 1962.

Scarf, F. L., Landau damping and the attenuation of whistlers, Phys. Fluids 5, 6-13, 1962.

Sonnerup, B. U. O., and M. J. Laird, On magnetospheric interchange instability, J. Geophys. Res., 68, 131-139, 1963.

Taylor, J. B., Equilibrium and stability of plasma in arbitrary mirror fields, Phys. Fluids 7, 767-773, 1964.

REFERENCES

Van Zandt, T. E., and R. W. Knecht, The structure and physics of the upper atmosphere, Chp. 6 in Space Physics, ed. D. P. LeGalley and A. Rosen, John Wiley and Sons, New York, 1964.

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PART 3

ON THE EFFECT OF RESONANT MAGNETIC-MOMENT VIOLATION ON TRAPPED PARTICLES

by

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ABSTRACT

1141

A quasilinear diffusion equation for particles interacting with transverse waves propagating along a homogeneous magnetic field is constructed to study the effect of resonant magnetic-moment-violating interactions. It appears that quite anisotropic flat-helix particle distributions can result from such interactions. Author

INTRODUCTION

It has been suggested that resonant particle-wave interactions in which the magnetic moment of the particle is violated, may be of importance in determining the loss of high energy particles from the radiation belts. Wentzel [1961] and Dragt [1961] have studied the resonant scattering of radiation belt protons by hydromagnetic waves, and have concluded that in some regions considerable scattering of the high energy protons could occur in times of the order of a day. Similiarly, the effect of whistlers on electrons has been analyzed by Dungey [1963] and by Cornwall [1964], and they also conclude that the resonant interaction may be an important loss mechanism. The interaction has also been suggested as an acceleration mechanism: Dessler [1962], for instance, proposed that Jovian whistlers might be responsible for the flat-helix relativistic electrons which exist in Jupiter's radiation belt if Jupiter's decimeter radiation is due to synchrotron radiation from electrons trapped in a dipole field [Chang and Davis (1962)]. Because of these various possibilities it is of interest to study in more detail the consequences of the resonant magnetic-moment-violating particle-wave interaction. Accordingly, in the following we use the quasilinear relativistic collisionless Boltzmann equation to study the resulting diffusion in energy and helix-angle. The results may be of significance for electron-whistler interactions and for proton-hydromagnetic wave interactions, as it is found that quite anisotropic distributions of the high energy particles may result.

QUASILINEAR DIFFUSION EQUATION

The relativistic collisionless Boltzmann equation will be used here to describe the particle distribution which results when electromagnetic waves propagate through a plasma. Denoting by $F^i(\vec{r}, \vec{p}, t)\, d^3r d^3p$ the number of particles of the i^{th} species in the position interval $d\vec{r}$ at $\vec{r}$ and the momentum interval $d\vec{p}$ at $\vec{p}$ at time t, we have

$$\frac{\partial F^i}{\partial t} + \frac{c\vec{p}\cdot \nabla F^i}{(m_i^2c^2+p^2)^{\frac{1}{2}}} + e_i\left[\vec{\mathcal{E}} + \frac{\vec{p}\times\vec{\mathcal{B}}}{(m_i^2c^2+p^2)^{\frac{1}{2}}}\right]\cdot \nabla_p F^i = 0 \quad . \tag{1}$$

Here $\vec{\mathcal{E}}(\vec{r}, t)$ and $\vec{\mathcal{B}}(\vec{r}, t)$ are the electric and magnetic fields, e_i and m_i are the charge and mass of the i^{th}-species particle, and c is the speed of light.

We wish to solve this equation in the quasilinear approximation developed by Drummond and Pines (1962) and by Vedenov, Velikhov and Sagdeev (1962) for treating the Vlasov equation. In this approximation some results of the solution to the linearized form of Eq. (1) are needed; so we shall begin by outlining the linear solution, referring for a more complete description to Chang (1963).

(1) Linearized equations

Writing $F^i(\vec{r}, \vec{p}, t) = F^i_o(\vec{p}) + F^i_1(\vec{r}, \vec{p}, t)$, where the equilibrium distribution density $F^i_o(\vec{p})$ is taken independent of position and $F^i_1(\vec{r}, \vec{p}, t)$ is considered to be a small perturbation, the zero and first-order equations are

$$\frac{e_i \vec{p} \times \vec{\mathcal{B}}_o \cdot \nabla_p F^i_o}{(m_i^2 c^2 + p^2)^{\frac{1}{2}}} = 0 \tag{2}$$

and

$$\frac{\partial F^i_1}{\partial t} + \frac{c\vec{p} \cdot \nabla F^i_1}{(m_i^2 c^2 + p^2)^{\frac{1}{2}}} + e_i \frac{\vec{p} \times \vec{\mathcal{B}}_o \cdot \nabla_p F^i_1}{(m_i^2 c^2 + p^2)^{\frac{1}{2}}} = - e_i \left[\vec{\mathcal{E}} + \frac{\vec{p} \times \vec{\mathcal{B}}_1}{(m_i^2 c^2 + p^2)^{\frac{1}{2}}} \right] \cdot \nabla_p F^i_o . \tag{3}$$

In these equations, $\vec{\mathcal{E}}(\vec{r}, t)$ has been considered a first-order quantity, and the magnetic field $\vec{\mathcal{B}}(\vec{r}, t)$ has been set equal to a large constant field $\vec{\mathcal{B}}_o$ plus a small perturbation $\vec{\mathcal{B}}_1(\vec{r}, t)$. The equation for the Fourier components

$$\{ f^i_1(\vec{k}, \vec{p}, \omega), \ \vec{E}(\vec{k}, \omega), \vec{B}_1(\vec{k}, \omega) \} =$$

$$\left(\frac{1}{2L}\right)^3 \int_{-L}^{L} d\vec{r}^3 \, dt \, \exp\left[-i(\vec{k} \cdot \vec{r} + \omega t)\right] \ \{ F^i_1(\vec{r}, \vec{p}, t), \ \vec{\mathcal{E}}(\vec{r}, t), \vec{\mathcal{B}}_1(\vec{r}, t) \} \tag{4}$$

$$\vec{k} = \frac{2\pi}{L} \vec{n} \qquad \vec{n} = \vec{1}, \ \vec{2},$$

which corresponds to Eq. (3) is

$$i\left[\omega+\frac{c\vec{k}.\vec{p}}{(m_i^2c^2+p^2)^{\frac{1}{2}}}\right]f_1^i+e_i\,\frac{\vec{p}\times\vec{\mathcal{B}}_o.\nabla_p f_1^i}{(m_i^2c^2+p^2)^{\frac{1}{2}}}=-e_i\left[\vec{E}+\frac{\vec{p}\times\vec{\mathcal{B}}_1}{(m_i^2c^2+p^2)^{\frac{1}{2}}}\right].\nabla_p F_o^i\,. \tag{5}$$

A dispersion formula is obtained from Eq. (5) by using Maxwell's equations to replace $\vec{B}_1$ by

$$\vec{B}_1=-\frac{c\vec{k}\times\vec{E}}{\omega} \tag{6}$$

and by forming the current fluctuation

$$\vec{j}(\omega,\vec{k})=\overleftrightarrow{\sigma}.\vec{E}=\sum_i e_i\int d\vec{p}^3\,\frac{\vec{p}}{(m_i^2c^2+p^2)^{\frac{1}{2}}}\,f_1(\vec{k},\vec{p},\omega)\;, \tag{7}$$

where the sum is over the different species. This expression combined with Maxwell's equations gives

$$\vec{k}\times(\vec{k}\times\vec{E})+\frac{\omega^2}{c^2}\vec{E}-i\,\frac{4\pi\omega}{c}\,\overleftrightarrow{\sigma}.\vec{E}=0 \tag{8}$$

from which the dispersion formula follows as

$$\left|k_\alpha k_\beta+\left(\frac{\omega^2}{c^2}-k^2\right)\delta_{\alpha\beta}-i\,\frac{4\pi\omega}{c}\,\sigma_{\alpha\beta}\right|=0\;. \tag{9}$$

If $\vec{\mathcal{B}}_o$ is taken to lie along the z-axis, $\vec{\mathcal{B}}_o=\mathcal{B}_o\vec{i}_z$; $\vec{k}$ is taken to lie in the x-z plane, $\vec{k}=k_\perp\vec{i}_z+k_\parallel\vec{i}_z$; and $\vec{p}$ is expressed in spherical coordinates

(p, θ, ϕ) with $\theta = 0$ denoting the direction of $\vec{i}_z$, Eqs. (5) and (7) give

$$\sigma_{\alpha\beta} = -\sum_i 2\pi e_i \int_0^{\pi}\int_0^{\infty} dp\,d\theta \, \frac{p^3 \sin\theta}{(m_i^2 c^2 + p^2)^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \frac{O^i_{\alpha\beta}(n)}{n + a_i} \tag{10}$$

where

$$O^i_{\alpha\beta}(n) = \begin{pmatrix} iA^i \sin\theta \left(\frac{nJ_n}{b_i}\right)^2 & A^i \sin\theta \left(\frac{n}{b_i} J_n J_n'\right) & iD^i \sin\theta \left(\frac{n}{b_i} J_n^2\right) \\ -A^i \sin\theta \left(\frac{n}{b_i} J_n J_n'\right) & iA^i \sin\theta (J_n')^2 & -D^i \sin\theta (J_n J_n') \\ iA^i \cos\theta \left(\frac{n}{b_i} J_n^2\right) & A^i \cos\theta (J_n J_n') & iD^i \cos\theta (J_n^2) \end{pmatrix}$$

and

$$a_i = -\frac{1}{e_i \mathcal{B}_o}\left[(m_i^2 c^2 + p^2)^{\frac{1}{2}} \omega + k_{\parallel} cp \cos\theta\right]$$

$$b = -\frac{k_{\perp} pc}{e_i \mathcal{B}_o} \sin\theta$$

$$A^i(p, \theta) = \frac{(m_i^2 c^2 + p^2)^{\frac{1}{2}}}{\mathcal{B}_o} \sin\theta \frac{\partial F_o^i}{\partial p} + \left[\frac{(m_i^2 c^2 + p^2)^{\frac{1}{2}}}{p \mathcal{B}_o} \cos\theta + \frac{k_{\parallel} c}{\mathcal{B}_o \omega}\right] \frac{\partial F_o^i}{\partial \theta} ,$$

$$D_i(p,\theta) = \frac{(m_i^2c^2+p^2)^{\frac{1}{2}}}{\mathcal{B}_o} \cos\theta \frac{\partial F_o^i}{\partial p} - \left[\frac{(m_i^2c^2+p^2)^{\frac{1}{2}}}{p\mathcal{B}_o}\sin\theta - \frac{ne_i}{\omega p\sin\theta}\right]\frac{\partial F_o^i}{\partial\theta} .$$

In this expression J_n and J_n' denote the n^{th} order Bessel function and its derivative, and the argument of the Bessel functions is the quantity b . From causality the residues at the singularities $n+a_i = 0$ are to be evaluated by considering [Landau (1949)]

$$\omega = \lim(\omega - i\nu) \quad \text{as} \quad \nu \to 0, \quad \nu > 0 \quad .$$

In the following, we shall restrict our attention to propagation along the magnetic field ($k_\perp = 0$), in which case the dispersion formula simplifies considerably. Here the conductivity tensor reduces to a convenient closed expression, and moreover the dispersion formula separates into three factors corresponding to two circularly polarized transverse modes and one longitudinal mode. Our present interest is in the transverse modes: $\vec{E}$ and $\vec{B}_1$ are perpendicular to $\vec{\mathcal{B}}_o$, and Eqs. (9) and (10) give

$$k_\parallel^2 - (\omega^2/c^2) = -i\frac{4\pi\omega}{c}\sigma_\parallel \pm \frac{4\pi\omega}{c}\sigma_{12}$$

$$\sigma_\parallel = \sum_i i\pi e_i \int_0^\pi\int_0^\infty dp d\theta \frac{p^3 a_i A^i \sin^2\theta}{(m_i^2c^2+p^2)^{\frac{1}{2}}(1-a_i^2)} \qquad (11)$$

$$\sigma_{12} = -\sum_i e_i\pi \int_0^\pi\int_0^\infty dp d\theta \frac{p^3 A^i \sin^2\theta}{(m_i^2c^2+p^2)^{\frac{1}{2}}(1-a_i^2)}$$

where a_i and A^i are as defined in Eq. (10) with $k_\perp = 0$, and where the plus and minus signs correspond to the oppositely circularly polarized modes.

We are interested in using Eq. (11) to describe whistlers and Alfvén waves in the radiation belts. There we envisage a plasma most of which is thermal, but which also contains a relatively small density of relativistic particles. The real part of the index of refraction may then be taken to be determined essentially by the thermal component. If temperature effects are ignored completely, then Eq. (11) gives for the refractive index $N = |k_\parallel c/\omega|$

$$N^2 = 1 - \left(\frac{\omega_p^-}{\omega}\right)^2 \left[1 \pm \frac{\omega_c^-}{\omega}\right]^{-1} - \left(\frac{\omega_p^+}{\omega}\right)^2 \left[1 \mp \frac{\omega_c^+}{\omega}\right]^{-1}$$

$$= 1 - (\omega_p^-)^2 \left(1 + \frac{m_e}{m_I}\right) \left[\omega^2 - \omega_c^-\omega_c^+ \pm \omega(\omega_c^- - \omega_c^+)\right]^{-1}, \quad (12)$$

where the top sign applies to the extraordinary wave (i. e. to the wave in which the electric and magnetic vectors rotate in the same sense as the positive ions gyrate about the magnetic field lines), and the lower sign applies to the oppositely polarized ordinary wave. Here $\omega_p^- = (4\pi ne^2/m_e)^{\frac{1}{2}}$ and $\omega_p^+ = (4\pi ne^2/m_I)^{\frac{1}{2}}$ are the electron and ion plasma frequencies, $\omega_c^- = e\mathcal{B}_o/m_e c$ and $\omega_c^+ = e\mathcal{B}_o/m_I c$ are the (non-relativistic) electron and ion gyration frequencies, n is the electron density, e is the magnitude of the electron charge, and we have taken the ions to be protons. In the ordinary whistler mode when $\omega_c^+ << \omega << \omega_c^-$, Eq. (12) gives

$$N^2 \approx 1 + \frac{(\omega_p^-)^2}{\omega_c^- \omega} ,$$

which when $(\omega_p^-)^2/\omega_c \omega >> 1$ becomes simply

$$N^2 \approx \frac{(\omega_p^-)^2}{\omega_c^- \omega} . \tag{13}$$

For the Alfven wave occurring when $\omega << \omega_c^+$, Eq. (12) gives

$$N^2 \approx 1 + \frac{(\omega_p^-)^2}{\omega_c^- \omega_c^+} ,$$

and when $(\omega_p^-)^2/\omega_c^- \omega_c^+ >> 1$, this becomes

$$N^2 \approx \frac{4\pi n m_I c^2}{B^2} . \tag{14}$$

Although the details of the distribution functions have been ignored in determing Eq. (12) for the real part of the dispersion relation, they cannot be ignored in determining the Landau damping. For the ordinary whistler mode where the ions are ignored, for instance, the damping results from only those electrons which resonate with the wave. From Eq. (11) the imaginary part of ω for this whistler is determined by the imaginary part of $\left[- i \frac{4\pi\omega}{c} \sigma_{\parallel} - \frac{4\pi\omega}{c} \sigma_{12}\right]$: a first iterate solution for

$\gamma = \text{Im}\,\omega$ results on ignoring γ inside this bracketed expression, in which case the imaginary contribution comes only from the poles $1-a^{e}=0$. Evaluating these residues according to the Landau (1949) prescription, we find

$$\text{Im}\left[-i4\pi\omega\frac{\sigma_{\parallel}}{c} \pm \frac{4\pi\omega}{c}\sigma_{12}\right] = \frac{4\pi^3\omega e^2}{c^2|k_{\parallel}|} I(p_1, p_2)$$

where

$$I(p, p_2) = \int_{p}^{p_2} dp\,p^2 \frac{\sin\theta}{(m_e^2c^2+p^2)^{\frac{1}{2}}}\left\{(m_e^2c^2+p^2)^{\frac{1}{2}}\sin\theta\frac{\partial F_o}{\partial p} + \left[\frac{(m_e^2c^2+p^2)^{\frac{1}{2}}}{p}\cos\theta + \frac{k_{\parallel}c}{\omega}\right]\frac{\partial F_o}{\partial\theta}\right\}$$

$$\cos\theta = -\frac{\omega m_e}{k_{\parallel}p}\left[\left(1+\left(\frac{p}{m_e c}\right)^2\right)^{\frac{1}{2}} - \frac{\omega_c^-}{\omega}\right]$$

and p_1 and p_2 are the smallest and largest p, respectively, for which $|\cos\theta| < 1$. Since the rate at which the wave changes its energy is seen to depend so strongly on the details of the distribution function for the resonating particles, it is expected that the distribution of resonant particles might be strongly effected by the wave. To see this effect, it is necessary to go beyond the linearized treatment.

(2) Quasilinear diffusion equation

In the quasilinear treatment of the Vlasov equation developed by Drummond and Pines (1962) and by Vedenov, Velikhov and Sagdeev (1962), a diffusion-like equation is obtained for the $\vec{k}=0$ Fourier component by writing the $\vec{k}=0$ convolution of the acceleration term through second order in the perturbation field. Thus, taking the $\vec{k}=0$ Fourier transform of Eq. (1), identifying the $\vec{k}=0$ transform of $F^i(\vec{r}, t)$ with F_o^i, and using Eq. (2) for the $k_\perp=0$ waves under study, we have the quasilinear diffusion equation

$$\frac{\partial F_o^i}{\partial t} + e_i \sum \left[\vec{E}(k_\parallel, \omega_{k_\parallel}) + \frac{\underline{p} \times \vec{B}_1(k_\parallel, \omega_{k_\parallel})}{(m_i^2 c^2 + p^2)^{\frac{1}{2}}} \right] \cdot {}_p f_1^i(-k_\parallel, \omega_{-k_\parallel}) = 0 \ . \qquad (16)$$

Here, $f_1^i(-k_\parallel, \omega_{-k_\parallel})$ is the solution of Eq. (5) for $\vec{k}=-k_\parallel \vec{i}_z$ and $\omega = \omega_{-k_\parallel}$; and the subscript $k_\parallel$ on $\omega_{k_\parallel}$ denotes the solution $\omega = \omega(k_\parallel)$ of the dispersion relation of Eq. (11). The convolution which forms the second term of Eq. (16) contains no oscillatory time factor, since for $\vec{E}(k_\parallel, \omega_{k_\parallel})$ and $\vec{E}(-k_\parallel, \omega_{-k_\parallel})$ to describe the same (real) wave we must have

$$E(-k_\parallel, \omega_{-k_\parallel}) = E^*(k_\parallel, \omega_{k_\parallel})$$

$$\mathcal{Re}\,(\omega_{-k_\parallel}) = -\,\mathcal{Re}\,(\omega_{k_\parallel}) \quad . \qquad (17)$$

$$\gamma_{-k_\parallel} \equiv \text{Im}\,(\omega_{-k_\parallel}) = \text{Im}\,(\omega_{k_\parallel}) \equiv \gamma_{k_\parallel} \ .$$

Writing

$$\vec{p} = p_\perp \cos\phi\ \vec{i}_x + p_\perp \sin\phi\ \vec{i}_y + p_\parallel \vec{i}_z \ ,$$

$$\vec{E} = E_x \vec{i}_x + E_y \vec{i}_y \ ,$$

Eq. (5) for the $f_1^i(k_\parallel, \omega)$ induced by the transverse waves with $k_\perp = 0$ may be written

$$\frac{\partial f_1^i}{\partial \phi} - \frac{i}{e_i \mathcal{B}_o} (m_i^2 c^2 + p^2)^{\frac{1}{2}} \left[\omega + \frac{p_\parallel k_\parallel c}{(m_i^2 c^2 + p^2)^{\frac{1}{2}}}\right] f_1^i =$$

$$\frac{(m_i^2 c^2 + p^2)^{\frac{1}{2}}}{\mathcal{B}_o} \left[\frac{\partial F_o}{\partial p_\perp} - \frac{k_\parallel c}{(m_i^2 c^2 + p^2)^{\frac{1}{2}} \omega} \left(p_\perp \frac{\partial F_o}{\partial p_\parallel} - p_\parallel \frac{\partial F_o}{\partial p_\perp}\right)\right] \Big[E_x \cos\phi$$

$$+ E_y \sin\phi\Big] \ . \tag{5a}$$

This has the solution periodic in ϕ:

$$f_1 = \left[E_x \cos\phi + E_y \sin\phi\right] \frac{A^i i a_i}{1 - a_i^2} + \left[E_x \sin\phi - E_y \cos\phi\right] \frac{A^i}{1 - a_i^2} \tag{18}$$

where A^i and a_i are as defined in Eq. (10) and which in terms of $p_\perp$ and $p_\parallel$ are

$$A^i = \frac{(m_i^2 c^2 + p^2)^{\frac{1}{2}}}{\mathcal{B}_o}\left[\frac{\partial F_o^i}{\partial p_\perp} - \frac{k_\parallel c}{(m_i^2 c^2 + p^2)^{\frac{1}{2}}\omega}\left(p_\perp \frac{\partial F_o^i}{\partial p_\parallel} - p_\parallel \frac{\partial F_o^i}{\partial p_\perp}\right)\right] \tag{19}$$

$$a^i = -\frac{1}{e_i \mathcal{B}_o}\left[(m_i^2 c^2 + p^2)^{\frac{1}{2}}\omega + k_\parallel p_\parallel c\right] \quad .$$

In these coordinates,

$$\left[\vec{E}(k_\parallel, \omega_{k_\parallel}) + \frac{\vec{p}\times\vec{B}_1(k_\parallel, \omega_{k_\parallel})}{(m_i^2 c^2 + p^2)^{\frac{1}{2}}}\right]\cdot \nabla_p f_1(-k_\parallel, \omega_{-k_\parallel}) =$$

$$E_\phi(k_\parallel, \omega_{k_\parallel})\left[1 + \frac{p_\parallel k_\parallel c}{\omega_{k_\parallel}(m_i^2 c^2 + p^2)^{\frac{1}{2}}}\right]\frac{1}{p_\perp}\frac{\partial f_1}{\partial \phi}(-k_\parallel, \omega_{-k_\parallel})$$

$$- \frac{p_\perp k_\parallel c E_\perp(k_\parallel, \omega_{k_\parallel})}{\omega_{k_\parallel}(m_i^2 c^2 + p^2)^{\frac{1}{2}}}\frac{\partial f_1}{\partial p_\parallel}(-k_\parallel, \omega_{-k_\parallel})$$

$$+ \left[1 + \frac{p_\parallel k_\parallel c}{\omega_{k_\parallel}(m_i^2 c^2 + p^2)^{\frac{1}{2}}}\right]E_\perp(k_\parallel, \omega_{k_\parallel})\frac{\partial f_1}{\partial p_\perp}(-k_\parallel, \omega_{-k_\parallel}) \, , \tag{20}$$

where

$$E_\perp = E_x \cos\phi + E_y \sin\phi$$

$$E_\phi = -E_x \sin\phi + E_y \cos\phi \quad . \tag{21}$$

Using Eq. (20) and Eq. (18) for f_1, we may then write Eq. (16) as

$$\frac{\partial F_o^i}{\partial t} = -e_i \sum \Bigg\{ E_\phi(k_\parallel, \omega_{k_\parallel}) \left(1 + \frac{p_\parallel k_\parallel c}{\omega_{k_\parallel}(m_i^2 c^2 + p^2)^{\frac{1}{2}}}\right) \frac{1}{p_\perp} \left[\frac{A^i ia_i}{(1-a_i^2)} \frac{\partial E_\perp}{\partial \phi} - \frac{A^i}{1-a_i^2} \frac{\partial E_\phi}{\partial \phi}\right]_{-k_\parallel,\ \omega_{-k_\parallel}}$$

$$+ E_\perp(k_\parallel, \omega_{k_\parallel}) \left(- \frac{p_\perp k_\parallel c}{\omega_{k_\parallel}(m_i^2 c^2 + p^2)^{\frac{1}{2}}}\right) \left[\frac{\partial}{\partial p_\parallel} \frac{A^i ia_i}{1-a_i^2} E_\perp - \frac{\partial}{\partial p_\parallel} \frac{A^i}{1-a_i^2} E_\phi\right]_{-k_\parallel,\ \omega_{-k_\parallel}}$$

$$+ E_\perp(k_\parallel, \omega_{k_\parallel}) \left(1 + \frac{p_\parallel k_\parallel c}{\omega_{k_\parallel}(m_i^2 c^2 + p^2)^{\frac{1}{2}}}\right) \left[\frac{\partial}{\partial p_\perp} \left(\frac{A^i ia_i}{1-a_i^2}\right) E_\perp - \frac{\partial}{\partial p_\perp} \left(\frac{A^i}{1-a_i^2}\right) E_\phi\right]_{-k_\parallel,\ \omega_{-k_\parallel}} \Bigg\} \tag{22}$$

where the subscripts $(-k_\parallel, \omega_{k_\parallel})$ on the square brackets denote the arguments of the enclosed quantities. Noting that

$$\frac{\partial E_\perp}{\partial \phi} = E_\phi \text{ and } \frac{\partial E_\phi}{\partial \phi} = - E_\perp , \tag{23}$$

this may be rewritten as

$$
\begin{aligned}
\frac{\partial F_o^i}{\partial t} = -e_i \sum \Bigg\{ & E_\phi(k_\parallel, \omega_{k_\parallel}) \left(1 + \frac{p_\parallel k_\parallel c}{\omega_{k_\parallel}(m_i^2 c^2 + p^2)^{\frac{1}{2}}}\right) \frac{1}{p_\perp} \left[\left(\frac{A^i i a_i}{1-a_i^2}\right) E_\phi + \left(\frac{A^i}{1-a_i^2}\right) E_\perp \right]_{-k_\parallel,\, \omega_{-k_\parallel}} \\
& + E_\perp(k_\parallel, \omega_{k_\parallel}) \left(- \frac{p_\perp k_\parallel c}{\omega_{k_\parallel}(m_i^2 c^2 + p^2)^{\frac{1}{2}}}\right) \left[\frac{\partial}{\partial p_\parallel} \left(\frac{A^i i a_i}{1-a_i^2}\right) E_\perp - \frac{\partial}{\partial p_\parallel} \left(\frac{A^i}{1-a_i^2}\right) E_\phi \right]_{-k_\parallel,\, \omega_{-k_\parallel}} \\
& + E_\perp(k_\parallel, \omega_{k_\parallel}) \left(1 + \frac{p_\parallel k_\parallel c}{\omega_{k_\parallel}(m_i^2 c^2 + p^2)^{\frac{1}{2}}}\right) \left[\frac{\partial}{\partial p_\perp} \left(\frac{A^i i a_i}{1-a_i^2}\right) E_\perp - \frac{\partial}{\partial p_\perp} \left(\frac{A^i}{1-a_i^2}\right) E_\phi \right]_{-k_\parallel,\, \omega_{-k_\parallel}} \Bigg\} .
\end{aligned}
\tag{24}
$$

For the circularly polarized modes of interest,

$$
E_y(k_\parallel, \omega_{k_\parallel}) = \pm i E_x(k_\parallel, \omega_{k_\parallel}) \quad \text{and} \quad E_y(-k_\parallel, \omega_{-k_\parallel}) = \mp i E_x(-k_\parallel, \omega_{-k_\parallel}) \tag{25}
$$

where the plus and minus signs refer to the extraordinary and ordinary modes, respectively; so that writing

$$
E_x(k_\parallel, \omega_{k_\parallel}) = E_{k_\parallel} \frac{1}{\sqrt{2}} , \tag{26}
$$

Eq. (24) may be further simplified here to

$$\frac{\partial F_o^i}{\partial t} = \mp\frac{e_i}{2}\sum\left\{\left(1+\frac{p_\parallel k_\parallel c}{\omega_{k_\parallel}(m_i^2c^2+p^2)^{\frac{1}{2}}}\right)\frac{1}{p_\perp}\,|E_{k_\parallel}|^2\left(\frac{iA^i}{1\mp a_i}\right)_{-k_\parallel,\,\omega_{-k_\parallel}}\right.$$

$$-\left(\frac{p_\perp k_\parallel c}{\omega_{k_\parallel}(m_i^2c^2+p^2)^{\frac{1}{2}}}\right)|E_{k_\parallel}|^2\,\frac{\partial}{\partial p_\parallel}\left(\frac{iA^i}{1\mp a_i}\right)_{-k_\parallel,\,\omega_{-k_\parallel}} \tag{27}$$

$$\left.+\left(1+\frac{p_\parallel k_\parallel c}{\omega_{k_\parallel}(m_i^2c^2+p^2)^{\frac{1}{2}}}\right)|E_{k_\parallel}|^2\,\frac{\partial}{\partial p_\perp}\left(\frac{iA^i}{1\mp a_i}\right)_{-k_\parallel,\,\omega_{-k_\parallel}}\right\}.$$

Reintroducting spherical coordinates,

$$p_\parallel = p\cos\theta,\quad p_\perp = p\sin\theta\ ,$$

Eq. (25) becomes

$$\frac{\partial F_o^i}{\partial t} = \mp\frac{e_i}{2}\sum|E_{k_\parallel}|^2\left\{\frac{k_\parallel c}{\omega_{k_\parallel}(m_i^2c^2+p^2)^{\frac{1}{2}}\sin\theta}\,\frac{\partial}{\partial\theta}\left[\frac{iA^i\sin\theta}{1\mp a_i}\right]_{-k_\parallel,\,\omega_{-k_\parallel}}\right. \tag{28}$$

$$+\frac{\cos^2\theta}{p\sin\theta}\,\frac{\partial}{\partial\theta}\left[\frac{\sin\theta}{\cos\theta}\left(\frac{iA^i}{1\mp a_i}\right)\right]_{-k_\parallel,\,\omega_{-k_\parallel}}$$

$$\left.+\sin\theta\,\frac{\partial}{\partial p}\left[\frac{iA^i}{1\mp a_i}\right]_{-k_\parallel,\,\omega_{-k_\parallel}}\right\}.$$

Here the top signs apply to the extraordinary mode and the bottom signs apply to the ordinary mode.

To determine the rate at which energy is pumped into the particles form the quantity

$$\sum_i \frac{\partial}{\partial t}\int dp\,d\theta\,d\phi \sqrt{p^2c^2+m_i^2c^4}\; p^2 \sin\theta F_o^i = \frac{\partial}{\partial t} K$$

$$= \sum \mp \pi e_i \left|E_{k_\parallel}\right|^2 \int dp\,d\theta \sin^2\theta \sqrt{p^2c^2+m_i^2c^4}\; \frac{\partial}{\partial p}\, p^2 \left[\frac{iA^i}{1\mp a_i}\right]_{-k_\parallel,\, \omega_{-k_\parallel}} \tag{29}$$

which upon integration by parts becomes

$$\frac{\partial}{\partial t} K = \pm\, \pi \sum e_i \left|E_{k_\parallel}\right|^2 \int dp\,d\theta \frac{p^3c^2\sin^2\theta}{\sqrt{p^2c^2+m_i^2c^4}} \left[\frac{iA^i}{1\mp a_i}\right]_{-k_\parallel,\, \omega_{-k_\parallel}} . \tag{30}$$

While from Eq. (11) we have

$$k_\parallel^2 - \frac{\omega^2}{c^2} = \mp \sum e_i 4\pi^2 \omega \int dp\,d\theta \frac{p^3 A^i \sin^2\theta}{(p^2c^2+m_i^2c^4)^{1/2}} \frac{1}{1\pm a_i} . \tag{31}$$

Recall that in Eq. (30) the quantity in the square bracket is evaluated at $-k_\parallel$ whereas the analagous terms in Eq. (31) are evaluated at $+k_\parallel$ interchanging the significance of $\pm$ thus Eq. (30) can be written

$$\frac{\partial}{\partial t} K = -\frac{1}{2\pi}\sum_{k_\parallel} |E_{k_\parallel}|^2 \left(\frac{k_\parallel^2}{\omega} - \frac{\omega}{c^2}\right) iA^i = -\frac{1}{4\pi}\sum 2\gamma_{k_\parallel} (|E_{k_\parallel}|^2 + |B_{k_\parallel}|^2)$$

$$= -\frac{\partial}{\partial t}\frac{1}{8\pi}\int dx \{(B(x)^2 + |E(x)|^2\} \qquad . \qquad (32)$$

Now since only the real part of the r. h. s. of Eq. (29) contributes it is easily seen that for $\omega < \omega_c^i$ only the ions are concerned while for $\omega_c^i \ll \omega < \omega_i^c$ it is the electrons since in the former only extraordinary waves are considered whereas in the latter only the ordinary waves contribute. Hence depending upon the range of frequency there is energy transferred from the wave to the ions or the electrons as the case may be.

Since A^i contains first derivatives of F_o^i, Eq. (26) has the form of a diffusion equation. In the integral defining the diffusion coefficient, the integrand becomes very large for the resonant particles at the singularity $1 \mp a_i = 0$. At the singularity, $(1 \mp a_i)^{-1}$ is to be evaluated according to the Landau prescription so that at the singularity $(1 \mp a_i)^{-1}$ behaves like the delta-function $i\pi\delta(1 \mp a_i)$, where the upper sign corresponds to the ordinary mode and describes the resonant interaction of electrons with Whistler's while the lower sign of ions is analogously defined for the Alfveh wave ion interaction. Because of the singular nature of $(1 \mp a_i)^{-1}$, we shall simply replace $(1 \pm a_i)^{-1}$ in Eq. (26) by $i\pi\delta(1 \mp a_i)$: this corresponds to looking at the diffusion resulting from resonant interactions. The particle motion resulting from nonresonant interactions in the collisionless Boltzmann equation is not only much smaller, but also does not represent true diffusion. This has been shown by applying the Laplace transform technique to the equation in an initial-value problem [Drummond and Pines (1962)], and alternatively by constructing a Fokker-Planck-like equation to describe the interaction of an ensemble of particles with a group of damped waves Chang (1964).

The diffusion equation describing resonant interactions is, replacing the sum by an integral over k ($\Sigma_k \rightarrow \frac{L}{2\pi} \int dk_\parallel$),

$$\frac{\partial F_o^i}{\partial t} = \pm \frac{e_i L}{4} \int dk_\parallel \left| E_{k_\parallel} \right|^2 \left\{ \frac{k_\parallel c}{\omega_{k_\parallel} (m_i^2 c^2 + p^2)^{\frac{1}{2}} \sin\theta} \frac{\partial}{\partial\theta} \left[A^i \sin\theta \, \delta(1 \mp a_i) \right] \right.$$

$$+ \frac{\cos^2\theta}{p \sin\theta} \frac{\partial}{\partial\theta} \left[\frac{\sin\theta}{\cos\theta} A^i \delta(1 \mp a_i) \right] \tag{33}$$

$$\left. + \sin\theta \frac{\partial}{\partial p} \left[A^i \delta(1 \mp a_i) \right] \right\}$$

where the quantities in the square brackets may be evaluated at $(k_\parallel, \omega_{k_\parallel})$ since we have the relations of Eq. (17) and we use $\mathcal{R}\ell\, \omega_{k_\parallel}$ for $\omega_{k_\parallel}$ ignoring terms of $O(\mathrm{Im}\, \omega_{k_\parallel} / \mathcal{R}\ell\, \omega_{k_\parallel})$.

DISCUSSION

The nature of the solution to Eq. (27) are easily seen in the non-relativistic limit. Since the energetic particles in the Van Allen belt are only mildly relativistic, this is not too unreasonable a limit to consider. Equation (27) then reduces to

$$\frac{\partial F_o^i}{\partial t} = \pm\frac{\pi e_i}{2}\frac{L}{2\pi}\int dk_{\parallel}|E_{k_{\parallel}}|^2 \Bigg\{ \frac{k_{\parallel}}{\omega_{k_{\parallel}} m_i \sin\theta}\frac{\partial}{\partial\theta}\Bigg[\sin\theta\ \delta(1\mp a_i)\Bigg\{\frac{m_i c}{\mathcal{B}_o}\sin\theta\frac{\partial F_o^i}{\partial p}$$

$$+\left(\frac{m_i c}{p\mathcal{B}_o}\cos\theta + \frac{k_{\parallel} c}{\mathcal{B}_o\omega}\right)\frac{\partial F_o^i}{\partial\theta}\Bigg\}\Bigg]$$

$$+\ \frac{\cos^2\theta}{p\sin\theta}\frac{\partial}{\partial\theta}\Bigg[\frac{\sin\theta}{\cos\theta}\delta(1\mp a_i)\Bigg\{\frac{m_i c}{\mathcal{B}_o}\sin\theta\frac{\partial F_o^i}{\partial p}$$

$$+\left(\frac{m_i c}{p\mathcal{B}_o}\cos\theta + \frac{k_{\parallel} c}{\mathcal{B}_o\omega}\right)\frac{\partial F_o^i}{\partial\theta}\Bigg\}\Bigg]$$

$$+\ \sin\theta\frac{\partial}{\partial p}\Bigg[\delta(1\mp a_i)\Bigg\{\frac{m_i c}{\mathcal{B}_o}\sin\theta\frac{\partial F_o^i}{\partial p}$$

$$+\left(\frac{m_i c}{p\mathcal{B}_o}\cos\theta + \frac{k_{\parallel} c}{\mathcal{B}_o\omega}\right)\frac{\partial F_o^i}{\partial\theta}\Bigg\}\Bigg]\Bigg\}. \tag{34}$$

For the modes of interest, $\omega << \frac{e_i \mathcal{B}_o}{m_i c}$, so that with the delta function condition, ω may be ignored compared to $k_{\parallel} p_{\parallel}/m_i$. Moreover, regarding $\frac{\partial F}{\partial p}$ to be of the order of F/p, and noting that the ratio $m_i \omega / k_{\parallel} p \cos\theta$ is given by the delta function condition to be of the order of $\omega/(e_i \mathcal{B}_o / m_i c) << 1$, we shall also ignore energy changes associated with the $\partial/\partial p$ derivatives. In that case, Eq. (34) may be further approximated:

$$\frac{\partial F_o^i}{\partial t} \approx \frac{\pi e_i^2}{p m_i \sin\theta} \frac{\partial}{\partial\theta} \left[\left(\frac{k_{\parallel}^2 |E_{k_{\parallel}}|^2}{\omega_{k_{\parallel}}^2} \right)_{k_{\parallel} \pm \frac{e \mathcal{B}_o}{cp\cos\theta}} \frac{\sin\theta}{\cos\theta} \frac{\partial F_o^i}{\partial\theta} \right] \frac{L}{4\pi} . \qquad (35)$$

With $\omega_{k_{\parallel}} \propto k_{\parallel}^{\beta}$ and $|E_{k_{\parallel}}|^2 \propto k_{\parallel}^{-\alpha}$ (a spectrum suggested by lightning for whistlers [Dungey (1963)] and by turbulence for Alfvén waves) the quantity

$$\left(\frac{k_{\parallel}^2 |E_{k_{\parallel}}|^2}{\omega_{k_{\parallel}}^2} \right)_{k_{\parallel} = \pm \frac{e \mathcal{B}_o}{cp\cos\theta}}$$

is proportional to $k_{\parallel}^{2-2\beta-\alpha}$, i. e. to $(p\cos\theta)^{\alpha+2\beta-2}$. The steady state solution of Eq. (35) with this spectrum of waves is given by

$$\frac{\partial F}{\partial\theta} \propto \frac{1}{\sin\theta(\cos\theta)^{\alpha+2\beta-3}} , \qquad (36)$$

corresponding to a very flat-helix spectrum when $\alpha + 2\beta - 3$ is large.

The time evolution of the solution to Eq (35) is easy to follow in the flat helix approximation Thus, replacing $\sin\theta$ by 1 and $\cos\theta$ by $x = \frac{\pi}{2} - \theta$, Eq. (35) reads

$$\frac{\partial F_o^i}{\partial \tau} \approx c_1 \frac{\partial}{\partial x} \left[x^{\alpha + 2\beta - 3} \frac{\partial F}{\partial x} \right] , \tag{37}$$

where $c_1(p)$ contains the proportionality constants of $\omega_{k_\parallel} \propto k_\parallel^\beta$ and $|E_{k_\parallel}|^2 \propto k_\parallel^{-\alpha}$. Equation (37) is of the form

$$\frac{\partial F_o^i}{\partial \tau} = - g \frac{\partial}{\partial x} (x^a F) + \frac{\partial^2}{\partial x^2} (x^{a+1} F) \tag{38}$$

where

$$\tau = c_1(p) t$$

$$a = \alpha + 2\beta - 4$$

$$g = \alpha + 2\beta - 3 \quad . \tag{39}$$

As a representative problem we examine the response of the system to a source

$$F_o^i (x, t = 0) = G(p) \delta(x - x_o)$$

which leads to the solution

Eq. (38) has the solution

$$F(x,\tau) = \frac{G(p)}{|a-1|} \; \frac{x_o^{\frac{1-g}{2}} \, x^{\frac{g-2a-1}{2}}}{\tau}$$

$$\exp\left[-\frac{1}{\tau(a-1)^2}\left(\frac{1}{x_o^{a-1}} + \frac{1}{x^{a-1}}\right)\right] \times \tag{40}$$

$$I_q\left(\frac{2}{\tau(a-1)^2}\,(x\,x_o)^{\frac{1-a}{2}}\right)$$

with

$$q = \left|\frac{g-1}{a-1}\right| \; ,$$

when F is required to be finite at the origin and an absorbing boundary is placed at $x \to \infty$. This solution, the time evolution of which has been discussed in detail elsewhere [Parker (1961), Davis and Chang (1962)], may be adapted to the present problem by the use of images.

The above equation (Eq. (40)) describes a time evolution in which the peak of the distribution moves toward $x=0$ ($\theta = \frac{\pi}{2}$) while the distribution itself tends to flatten in such a manner as to leak most of the particles toward $x = \frac{\pi}{2}$ ($\theta = 0$); however these particles then find themselves in the loss cone and will be removed leading to a final distribution which tends to be peaked at $x = 0$ ($\theta = \frac{\pi}{2}$).

Now, since the distribution flattens in angle (except near $\theta = \pi/2$) at long times it is no longer correct to neglect the derivative with respect to p. Thus rather than the steady state solution given by Eq. (39) we must look for solutions of the full time independent equation Eq. (34). It is interesting to note that one such equation which is independent of the spectrum is obtained by setting

$$\frac{m_i c}{\beta_o} \sin\theta \frac{\partial F_o^i}{\partial p} + \left(\frac{m_i c}{p \beta_o} \cos\theta + \frac{k_\parallel c}{\beta_o \omega} \right) \frac{\partial F_o^i}{\partial \theta} \approx$$

$$\frac{c}{\beta_o} \left[m_i \sin\theta \frac{\partial F_o^i}{\partial p} + \frac{k_\parallel}{\omega_{k_\parallel}} \frac{\partial F_o^i}{\partial \theta} \right] = 0 \quad . \tag{42}$$

The solution to Eq (42) is any general function of the solution to the characteristic equation

$$\frac{dp}{d\theta} = \frac{m_i \omega_{k_\parallel}}{k_\parallel} \sin\theta \quad . \tag{43}$$

For the interaction of Alfvén waves with protons, Eq (14) and the fact that the particle and wave move in opposite directions for resonance from the Doppler shift give

$$\frac{\omega_{k_\parallel}}{k_\parallel} = \frac{B}{(4\pi n m_I)^{\frac{1}{2}}} \equiv v_{\text{Alfvén}} \tag{44}$$

when $p\cos\theta > 0$, so that in this case

$$F_o^I = f(p + m_i v_{\text{Alfvén}} \cos\theta) \quad . \tag{45}$$

O'Brien (1964) states that the energetic proton spectrum in the belt is described by the empirical relation of McIlwain and Pizzella (1963)

$$j(E_p)dE = \text{constant } e^{-E_p/E_o} dE \tag{46}$$

$$E_o = (306 \pm 28)\ L^{-(5.2 \pm 0.2)} \text{MeV for } 1.2 < L \lesssim 8$$

where $j(E_p)$ denotes the intensity of protons of energy E_p and L is the magnetic shell parameter. This suggests using $f(x) \propto e^{-c_1 x^2}$ in Eq. (45) in which case

$$F_o^I \propto \exp\left[-c_1(p + m_i v_{\text{Alfvén}} \cos\theta)^2\right], \tag{47}$$

i.e. a flat-helix distribution results. Similarily, for electrons, Eq. (13) and the condition for a Doppler-shifted resonance give

$$\frac{\omega_{k_\parallel}}{k_\parallel} = \frac{c^2 \omega_c^2 m_e}{\omega_p^2 p \cos\theta} \tag{48}$$

so that in this case

$$F_o^e = f\left(p^2 + \frac{2m_e^2 c^2 \omega_c^2}{\omega_p^2} \ln \cos \theta\right) \quad . \tag{49}$$

If we take $f(x) \propto e^{-c_2 x}$ this gives

$$F_o^e \propto (\cos \theta)^{-\nu} \exp(-c_2 p^2) \, , \tag{50}$$

$$\nu = \frac{2m_e^2 c^2 \omega_c^2 c_2}{\omega_p^2}$$

which suggest a very flat-helix distribution for the electrons which are energetic enough to have Doppler-shifted resonances.

To get an idea of the representative times involved consider the early time form of the diffusion equation Eq. (35) from which we extract the diffusion coefficient

$$\sin \theta \, \frac{L}{4\pi} \, \frac{\pi e^2}{p_\parallel m c^2} \, |B|^2_{k_\parallel} = \pm \frac{e\mathcal{B}_o}{c \, p \cos \theta} = D \tag{51}$$

writing

$$\int |B_{k_\parallel}|^2 \, dk_\parallel = |B_{k_{\parallel o}}|^2 \, \Delta k_\parallel$$

with

$$= \frac{2\pi}{L} \sum |B_{k_\parallel}|^2 = \frac{2\pi}{L} \int_{-L}^{L} |B(x)|^2 \frac{dx}{2L}$$

we have

$$D = \frac{\pi e^2}{2p_\parallel mc^2} \frac{\sin^2\theta}{\Delta k_\parallel} \int_{-L}^{L} |B(x)|^2 \frac{dx}{2L} \tag{52}$$

in arriving at this result we have assumed that the spectrum of waves was a relatively flat function of wave number, $k_\parallel$.

For the Whistler mode we have

$$\Delta k_\parallel = \frac{2\Delta\omega}{\sqrt{\omega}} \sqrt{\frac{\omega_p^2}{\omega_c}} \frac{1}{c}$$

while for Alfvén waves

$$\Delta k_\parallel = \Delta\omega \sqrt{\frac{4\pi n m_i}{B_o^2}} .$$

To obtain an estimate of the order of the effect we assume as Dungey (1963)

$$\int B^2(x) \frac{dx}{2} \sim N\ (\text{gamma})^2$$

$$\omega \approx 2 \times 10^4 \quad \Delta\omega \approx 10^4$$

where N is the fraction of time disturbance is present. Assume that there are ten Whistlers per day and that on the average each electron sees the wave ten times we have

$$N = 100 \frac{R}{v_{\parallel}} \times 10^{-5}$$

and

$$D = 10^{10} \sqrt{\frac{\beta_o}{n}} \; \frac{R}{v_{\parallel}^2}$$

hence with $v_{\parallel} \approx c$ the diffusion time is of the order of a day. Dungey and Cornwall have pointed out that on replacing $v_{\parallel}$ here by its resonance condition $v_{\parallel} \sim \omega_c / k_{\parallel}$, a rapid increase in D with increasing distance from the earth is obtained.

To estimate the effects of Alfvén waves we assume that waves are formed at the interface of the magnetosphere and the solar wind with an average energy on the order of the geomagnetic energy at 10 earth radii. To estimate the energy density of waves deposited deep into the magnetosphere we utilize the constancy of the energy flow $|B|^2 \frac{d\omega}{dk} 4\pi R^2$. In order to account for the large fraction of waves which are reflected we multiply the conserved quantity by some small efficiency factor (η). Hence at an arbitrary position in the belt we have $|B_k|^2 = 10^{-7} \left(\frac{R}{10}\right)^{7/2} \eta$ where we assume $V_{\text{Alfvén}} \sim \frac{1}{R^{3/2}}$ then

$$D \approx \frac{3}{\Delta\omega} \; \frac{V_{\text{Alfvén}}}{v_{\parallel}} \left(\frac{R}{10}\right)^{7/2}$$

thus with $\Delta\omega \sim 1$, V_{Alfven} v and $\eta \sim 10^{-3}$ we obtain at L = 1.5 a diffusion time of the order of a day. Once again by replacing $v_{\parallel}$ by its resonance relation we find a rapid increase of D with R a behavior previously demonstrated by Dragt and Wentzel.

These results have time scales which have been obtained by previous workers in references cited in the Introduction and are comparable to natural times in the belt. Thus, it appears that resonant interactions in which the magnetic moment adiabatic invariant is violated may provide a means of obtaining energetic anisotropic particle distributions. A more detailed treatment of the quasilinear diffusion equation in which longitudinal as well as transverse fields are present and in which arbitrary directions of wave propagation are included, would seem justified as more observational data on the spectrum of waves become available.

REFERENCES

Chang, D. B., Amplified whistlers as the source of Jupiter's sporadic decameter radiation, Astrophys. J. 138, 1231-1241, 1963.

Chang, D. B., Landau damping and related phenomena, to be published, Phys. Fluids.

Chang, D. B., and L. Davis, Jr., Synchrotron radiation as the source of Jupiter's polarized decameter radiation, Ap.J. 136, 567-581, 1962.

Cornwall, J. M., Scattering of energetic trapped electrons by very low frequency waves, J. Geophys. Res., 69, 1251-1258, 1964.

Davis, L., Jr. and D. B. Chang, On the effect of geomagnetic fluctuations on trapped particles, J. Geophys. Res., 67, 2169-2179, 1962.

Dessler, A. J., Hydromagnetic picture of earth's storms, J. Phys. Soc. Japan 17, Supp. A-II 13-15, 1962.

Dragt, A. J., Effect of hydromagnetic waves on the lifetime of Van Allen radiation protons, J. Geophys. Res. 66, 1641-1649, 1961.

Drummond, W. E. and D. Pines, Nonlinear stability of plasma oscillations, Nuclear Fusion, Suppl. 2, Part 3, 1049-1057, 1962.

Dungey, J. W., Loss of Van Allen electrons due to whistlers, Planet Space Sci., 11, 591-595, 1963

Landau, L., On the vibrations of the electronic plasma, J. of Phys., 10, 25-34, 1945.

McIlwain, C. E. and G. Pizzella, On the energy spectrum of protons trapped in the earth's inner Van Allen zone, J. Geophys. Res. 68, 1811-1823, 1963.

O'Brien, B. J., The trapped radiation zones, Chpt. 14 in Space Physics ed. D. P. LeGalley and A. Rosen, John Wiley and Sons, Inc., New York, 1964.

Vedenov, A. A., E. P. Velikhov and R. Z. Sagdeev, Quasilinear theory of plasma equations, Nuclear Fusion, Supplement Part 2, 465-475, 1962.

Wentzel, D. G., Hydromagnetic waves and trapped radiation, Part 1, Breakdown on the adiabatic invariance, J. Geophys. Res. 66, 359-362, 1961, Part 2, Displacement of the mirror points, J. Geophys. Res. 66, 363-369, 1961.

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PART 4

REMARKS ON AN EXACT SOLUTION OF A UNIVERSAL INSTABILITY

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To determine the growth rates and the stability criteria of the modes which are described by the term "universal instability" it has been previously necessary to resort to a local treatment relying on a WKB calculation to justify the local results[1-4]. We report herein on an exact solution to the governing equations in the drift approximation[1] which verifies the local or WKB treatment when the usual bounds are placed on the density gradient. We maintain a finite Debye length to produce, in the local limit, the "universal instability" in the drift approximation[3,4].

We have in mind an unperturbed density which approaches a constant value at $x = -\infty$ and a constant and smaller value at $x = +\infty$; such behavior can be described by

$$n = n_0\left(1 - \eta \tanh \frac{x}{2\lambda}\right), \qquad 0 < \eta < 1. \tag{1}$$

The perturbed potential satisfies the equation

$$\nabla^2\phi = \frac{1}{n_0\lambda_D^2}\left(1 - i\,\frac{\omega}{k_{\parallel}}\sqrt{\pi\alpha}\right)\left(n - \frac{k_{\perp}}{2\alpha\omega\Omega}\frac{\partial n}{\partial x}\right)\phi\ , \tag{2}$$

where

$$\phi = \phi(x)e^{i\omega t + ik_{\perp}y + ik_{\parallel}z}, \qquad k_{\perp} >> k_{\parallel} \tag{3}$$

and $\Omega = \left|\frac{eB}{mc}\right|$ where $\vec{B}$ is constant and in the z-direction. In deriving Eq. (2) we have assumed $\frac{\omega}{k_{\parallel}}\sqrt{2\alpha} << 0$ whereas $\frac{\omega}{k_{\parallel}}\sqrt{2\alpha_i} >> 1$, where $\sqrt{\frac{1}{2\alpha}}\left(\sqrt{\frac{1}{2\alpha_i}}\right)$ is the thermal velocity of the electrons (ions). Also in Eq. (2) λ_D is the electron Debye length at $x = 0$. If Eq. (1) is substituted into Eq. (2) we obtain

$$\frac{\partial^2\phi}{\partial x^2} = \left[k_\perp^2\lambda_D^2 + \left(1 - i\,\frac{\omega}{k_\parallel}\sqrt{\pi\alpha}\right)\left(-\frac{kv_c}{\omega}\,\mathrm{sech}^2\,\frac{x}{2\lambda} + \left(1 - \eta\tanh\frac{x}{2\lambda}\right)\right)\right]\phi$$

$$= Q(x)\phi \tag{4}$$

wherein we neglect $k_\parallel$ compared with $k_\perp$ and

$$kv_c = \frac{k_\perp}{2\alpha\Omega}\,\frac{1}{n}\,\frac{\partial n}{\partial x}\bigg)_{x=0} = \frac{k_\perp}{2\alpha\Omega}\,\frac{\eta}{2\lambda}\ . \tag{5}$$

With the substitution

$$\xi = -e^{x/\lambda}, \qquad \phi = \xi^s(1-\xi)^t\,\psi(\xi) \tag{6}$$

we obtain the equation

$$(1-\xi)\xi\psi'' + \left[2s+1 - (2s+1+t)\xi\right]\psi' + \left[\left(\frac{\lambda}{\lambda_D}\right)^2\left(2\eta - \frac{4kv_c}{\omega}\right)\left(1 - \frac{i\omega}{k_\parallel}\sqrt{\pi\alpha}\right) - (2s+1)t\right]\psi = 0 \tag{7}$$

if s and t satisfy the following relations

$$s = \frac{\lambda}{\lambda_D}\left[k_\perp^2\lambda_D^2 + \left(1 - i\,\frac{\omega}{k_\parallel}\sqrt{\pi\alpha}\right)(1-\eta)\right]^{1/2}$$

$$t = \frac{1}{2} - \left[\frac{1}{4} + \frac{4kv_c}{\omega}\left(1 - i\,\frac{\omega}{k_\parallel}\sqrt{\pi\alpha}\right)\left(\frac{\lambda}{\lambda_D}\right)^2\right]^{1/2} \tag{8}$$

The solution to Eq. (7) is a hypergeometric function.[6] Hence collecting all terms we have

$$\phi = (-1)^s\,e^{-sx/\lambda}\,\left(1 + e^{-x/\lambda}\right)^t\,F(a_-,a_+;c;-e^{-x/\lambda})\ , \tag{9}$$

where

$$a_\pm = s+t \pm \frac{\lambda}{\lambda_D}\left[k_\perp^2\lambda_D^2 + \left(1 - i\,\frac{\omega}{k_{\|}}\sqrt{\pi\alpha}\right)(1+\eta)\right]^{1/2}$$

$$c = 2s+1 \ . \qquad (10)$$

It is immediately obvious that ϕ is well behaved as $x \to +\infty$, whereas as $x \to -\infty$ we have

$$\phi \sim (-1)^s\Gamma(c)\left\{\frac{\Gamma(a_+-a_-)}{\Gamma(a_+)\Gamma(c-a_-)}\, e^{-\frac{x}{\lambda_D}\left[k_\perp^2\lambda_D^2 + \left(1 - i\frac{\omega}{k_{\|}}\sqrt{\pi\alpha}\right)(1+\eta)\right]^{1/2}} + \frac{\Gamma(a_--a_+)}{\Gamma(a_-)\Gamma(c-a_+)}\, e^{\frac{x}{\lambda_D}\left[k_\perp^2\lambda_D^2 + \left(1 - i\frac{\omega}{k_{\|}}\sqrt{\pi\alpha}\right)(1+\eta)\right]^{1/2}}\right\} \qquad (11)$$

Hence, for the solution to be well behaved in this limit the first expression in the curly bracket must vanish leading to the equation

$$\left[\frac{1}{4}\left(\frac{\lambda_D}{\lambda}\right)^2 + \frac{4kv_c}{\omega}\left(1 - i\,\frac{\omega}{k_{\|}}\sqrt{\pi\alpha}\right)\right]^{1/2} - \left[k_\perp^2\lambda_D^2 + \left(1 - i\,\frac{\omega}{k_{\|}}\sqrt{\pi\alpha}\right)(1+\eta)\right]^{1/2}$$
$$-\left[k_\perp^2\lambda_D^2 + \left(1 - i\,\frac{\omega}{k_{\|}}\sqrt{\pi\alpha}\right)(1-\eta)\right]^{1/2} = \left(n + \frac{1}{2}\right)\frac{\lambda_D}{\lambda}\ , \quad n = 0,1,2\ldots \qquad (12)$$

which is just the condition that the argument of the gamma function in the denominator be zero or a negative integer.

To see how this compares with the usual WKB treatment we evaluate the phase integral condition[2,3]

$$\int\sqrt{-Q}\,dx = \left(n + \frac{1}{2}\right)\pi \qquad (13)$$

where the endpoints of the integrand occur at $Q = 0$. We then have

$$\left(n + \frac{1}{2}\right)\pi = -\int \frac{d\xi}{1-\xi}\,\frac{1}{\xi}\left\{-\,\frac{a_+ - a_-}{2}\,\xi^2 + 2\xi\,\frac{\lambda}{\lambda_D}\left[k_\perp^2\lambda_D^2 + \left(1 - i\,\frac{\omega}{k_\parallel}\sqrt{\pi\alpha}\right)\left(1 - 2\,\frac{kv_c}{\omega}\right)\right] - s\right\}^{1/2} \tag{14}$$

With $\xi = -e^{-x/\lambda} < 0$ the above integral is of a standard form and can easily be evaluated to yield

$$\left[\frac{4kv_c}{\omega}\left(1 - i\,\frac{\omega}{k_\parallel}\sqrt{\pi\alpha}\right)\right]^{1/2} - \left[k_\perp^2\lambda_D^2 + \left(1 - i\,\frac{\omega}{k_\parallel}\sqrt{\pi\alpha}\right)(1 + \eta)\right]^{1/2} -$$

$$\left[k_\perp^2\lambda_D^2 + \left(1 - i\,\frac{\omega}{k_\parallel}\sqrt{\pi\alpha}\right)(1 - \eta)\right]^{1/2} = \left(n + \frac{1}{2}\right)\frac{\lambda_D}{\lambda} \qquad n = 0,1,2\ldots \tag{15}$$

The local result obtains by simultaneously solving $Q = \frac{dQ}{dx} = 0$, which is equivalent to Eq. (15) with the right hand side equal to zero. We see that the WKB and exact solution differ only by the term $\frac{1}{4}\left(\frac{\lambda_D}{\lambda}\right)^2$ occurring in the first square root of Eq. (12) whereas the local theory differs by a term of order λ_D/λ.

It should be remarked that n is limited in size since inherent in the drift approximation is the necessity that the effective wavelength in the direction of the density gradient be large compared with the ion gyro radius. Indeed, since $\lambda_D/\lambda \ll 1$, to produce sizeable corrections to the local theory n must be so large as to violate our basic assumption.

Consequently we may conclude that, provided the Debye length and the ion gyro radius are small compared with λ, the distance in which the density varies, the local, WKB and exact solution are equivalent.

REFERENCES

1. L. I. Rudakov and R. Z. Sagdeev, Soviet Phys. Dokl. 6, 415 (1961).
2. A. A. Galeyev, Soviet Phys. Dokl. 8, 444 (1964).
3. N. A. Krall and M. N. Rosenbluth, to be published in Physics of Fluids.
4. For a more complete list of references, see Ref. 3.
5. F. C. Hoh, Preprint.
6. W. Magnus and F. Oberhettinger, Formulas and Theorems for the Special Functions of Mathematical Physics (Chelsea Publishing Co., New York, New York).

PART 5

During the period November 16, 1964 to March 15, 1965, the following activities were pursued by program personnel:

Dr. L. B. Pearlstein

Delivered a colloquium at Goddard Air Force Base on November 24.

Presented a seminar at the University of California at San Diego on March 11.

Dr. D. B. Chang

Was a member of the Fellowship Selection Committee for the Danforth Foundation, St. Louis, Missouri, December 16 through 20.

Attended the DASA Trapped Radiation Meeting at Boulder, Colorado, December 9 through 11, and presented a paper.

Attended the American Geophysical Union Meeting at Seattle, Washington, December 28 through January 1.

Drs. Pearlstein and Chang attended the meeting on collisionless shocks at Ames Laboratory, Palo Alto, March 1 through 3.

During the period of this report, no "reportable items" as defined by the article "Reporting of New Technology" evolved.